CHAPTER 5

Binomial Model

The binomial model is a discrete time model, and it is the simplest possible nontrivial model of a financial market.

- Almost all important concepts which we will further study later on, have a most intuitive and accessible version in the binomial case
- The binomial model is often used in practice

The model is very easy to understand, the mathematics required to analyze it can be set at high school level. For good reasons, we will build it on probability theory reviewed in Chapter 2 and 3.

The building block is one-step binomial model, assume that a coin toss is driving the market: the stock price goes up from S_0 to $S_0 u$ with probability p (of getting a head), or goes down to $S_0 d$, for brevity, we denoted this one-step binomial model by $\mathrm{BM}(S_0, u, d, p, r)$ with r being the risk-free rate. For a T-step binomial model driven by a coin toss sequence, we denote it by $\mathrm{BM}_T(S_0, u, d, p, r)$.

In this text, a *claim* is the general term for financial assets. A *contingent claim* is another term for a derivative with a payoff that is dependent on the realization of some uncertain future event. By creating a right and not an obligation, the contingent claim acts as a form of insurance against counterparty risk. Any derivative instrument that isn't a contingent claim is called a forward commitment.

§ 5.1 One-step Binomial Model

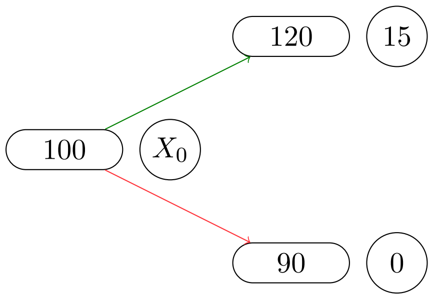
One period has two time points, we call the beginning of the period time zero and the end of the period time one. Free of arbitrage and completeness are most straightforward in the settings of one-step binomial model.

5.1.1 Price of a Call Option

We start by considering a very simple situation.

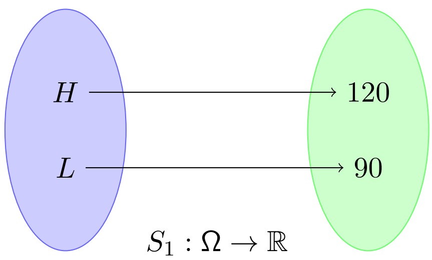
Example 5.1.1: We have a stock presently priced at \$100. In exactly one year the stock price will be either \$90 or \$120, with equal possibility. The current interest rate is 12.5% compounded yearly. What is the fair price for a European call option on the stock with a strike price \$105 expiring in one year?

Let the present time be $t = 0$, then the stock price is $S_0 = 100$. For the risk-free bond, $B_0 = 1$, let the continuous compounding interest rate be r, then $B_1 = e^r = 1.125$. At the future time $t = 1$, the payoffs of stock and option are depicted in the following figure:



To capture the two states of stock price movements, we assume that the random source is a coin toss, $\Omega = \{H, L\}$, with H and L standing for head and tail respectively. At $t = 1$, the stock price is S_1, with $S_1(H) = 120$ and $S_1(L) = 90$. The payoff of the option is X, with $X(H) = 15$ and $X(L) = 0$. We are interest in valuing $X_0 = \wp(X)$, the price of the call option.

Remark: At $t = 0$, $S_1(H) = 120$ and $S_1(L) = 90$ are known numbers, but S_1 is a random variable. More exactly, S_1 is a function that maps $\{H\}$ to 120 and $\{L\}$ to 90:



We are not totally blind to random variables, we do not know nothing at all. We do know something, $S_1(H)$ and $S_1(L)$ are numbers given, but which one will be taken does not known at $t = 0$. Only at

$t = 1$, the state H or L is revealed, then the (function) value of S_1 is determined accordingly.

A: Replication

Replicating imitates a given asset, usually a financial derivative, by searching for a portfolio of traded assets. Let us consider the following portfolio V:

- Long: 0.5 shares of stock
- Short: 40 bonds

In other words, to form portfolio V, we buy 0.5 shares of stock, and borrow \$40. By law of linear combination, the price of portfolio V is $V_0 = 0.5S_0 - 40B_0 = 10$, the value of portfolio V at time $t = 1$ is

$$V = 0.5S_1 - 40B_1$$

In more details, if the stock goes up

$$V(H) = 0.5S_1(H) - 40B_1(H) = 0.5 \cdot 120 + (-40) \cdot 1.125 = 15 = X(H)$$

If the stock goes down

$$V(L) = 0.5S_1(L) - 40B_1(L) = 0.5 \cdot 90 + (-40) \cdot 1.125 = 0 = X(L)$$

We see that the payoff of the option is replicated, $X = V$, portfolio V is called the *replicating portfolio* of the option. Thus, the price of the option is

$$X_0 = \wp(X) = \wp(V) = V_0 = 10$$

B: Hedge

In a perfect hedge, two investments were combined to produce a fixed return such that the risks are completely offset. In this sense, hedging is the replication of the risk-free asset. Let us create the following hedging portfolio V:

- Long: 0.5 shares of stock
- Short: 1 option

By law of linear combination, the value of portfolio V at time $t = 1$ is $V = 0.5S_1 - X$: In more details, if it is in the up state

$$V(H) = 0.5S_1(H) - X(H) = 0.5 \cdot 120 - 15 = 45$$

and if it is in the down state

$$V(L) = 0.5S_1(L) - X(L) = 0.5 \cdot 90 - 0 = 45$$

the value remains constant, it is a risk-free asset. Thus

$$V_0 = \wp(V) = \wp(45) = 45\wp(1) = 45e^{-r} = 40$$

Since $V_0 = 0.5S_0 - X_0$, we have $X_0 = 0.5S_0 - V_0 = 10$.

Remark: Before we know a good theory, we may know some working ways to tackle the real world problems.

5.1.2 Model Setup

We now discuss a simple one-step binomial model in which we are interested in valuing an option on the stock (a European call option, or any other derivative dependent on the stock).

A: Market Setting

We have two points in time, $t = 0$ ("today") and $t = 1$ ("tomorrow", next year, etc). In the model we have two assets: a risk-free bond (risk-free rate at constant r per period) and a stock. At time t the price of a bond is denoted by

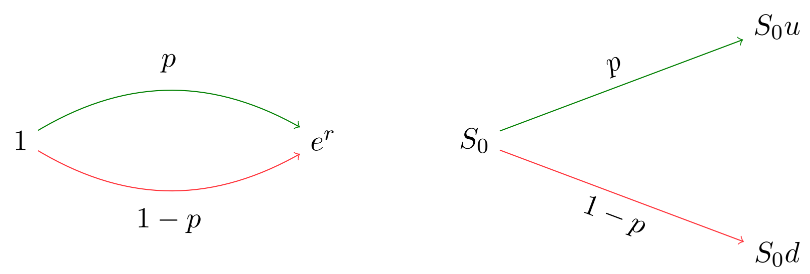
$$B_t = B(t) = e^{rt} \qquad t = 0, 1$$

and the price of one share of the stock is denoted by $S_t = S(t)$. Today's stock price is S_0, the future price of stock S_1 is described as follows:

$$S_1 = \begin{cases} S_0 u & p \\ S_0 d & 1 - p \end{cases}$$

Figure 5.1 shows the price processes of the bond and the stock.

Figure 5.1: Binomial Model There are two assets: A risk-free bond at price $B_0 = 1$ and a stock at price S_0. At the future time $t = 1$, the bond's price is $B_1 = e^r$, and the price of stock S_1 is a random variable, taking value $S_1(H) = S_0 u$ with probability p and $S_1(L) = S_0 d$ with probability $1 - p$.



Let the market be driven by a coin toss, we denote the sample space $\Omega = \{H, L\}$, where H standing for head and L for tail. The coin is not necessary fair, with probability $\mathrm{P}(H) = p$, $\mathrm{P}(L) = 1 - p$. Let the event space be $\mathcal{F} = \{\emptyset, \Omega, \{H\}, \{L\}\}$, then $(\Omega, \mathcal{F}, \mathrm{P})$ is the probability space for binomial model. Define the up-and-down random variable Z as

$$Z = \begin{cases} u & p \\ d & 1 - p \end{cases} \tag{5.1}$$

then, $Z(H) = u$ and $Z(L) = d$, Z is the random source of the market, and

$$S_1 = S_0 Z$$

We assume that today's stock price S_0 is known at time 0, as are the positive constants u, d, and p, with $d < u$. For brevity, the market setting of one-step binomial model will be denoted by $\mathrm{BM}(S_0, u, d, p, r)$.

B: Free of Arbitrage

We will study the behavior of various portfolios in the $\mathrm{BM}(S_0, u, d, p, r)$ market, and to this end we define a *portfolio* as a vector

$$\mathbf{h} = \begin{bmatrix} a \\ b \end{bmatrix} = [a; b]$$

The interpretation is that a is the number of shares of the stock held by us, whereas b is the number of risk-free bonds we hold in our portfolio. In one-step binomial model, the marketable payoff space is

$$\mathsf{X} = \{aS_1 + bB_1 : a, b \in \mathbb{R}\} = \mathrm{sp}(S_1, B_1)$$

Remark: Following the perfect market assumption, it is quite acceptable for a and b to be positive as well as negative. If, for example, $b = 3$, this means that we have bought three bonds at time $t = 0$. If on the other hand $a = -2$, this means that we have sold two shares of the stock at time $t = 0$, we borrow two shares and pay back at time 1. In financial jargon we have a long position in 3 bonds and a short position in 2 shares of the stock. It is an important assumption of the model that short positions are allowed.

Definition 5.1: Let $\mathbf{x}_t = [S_t; B_t]$, the *value process* of portfolio $\mathbf{h} = [a; b]$ is defined by

$$V_t = \mathbf{x}_t'\mathbf{h} = aS_t + bB_t \qquad t = 0, 1$$

or, in more detail

$$V_0 = b + aS_0$$
$$V_1 = be^r + aS_0Z$$

At time $t = 1$, please note that we do not rebalance the portfolio, the portfolio is formed at $t = 0$, and hold to $t = 1$. When the new price reveal, the value of the portfolio change.

By the definition of an arbitrage opportunity (Condition 1.13 on page 23), in one-step binomial model, an *arbitrage* portfolio is a portfolio with the properties (the symbol "$\gneqq$", weak greater than, is introduced in page 18 of §1.2.4)

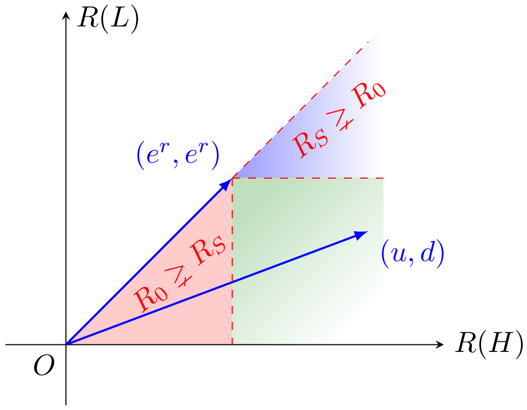
$$V_0 = 0, V_1 \gneqq 0$$

A *strong arbitrage* portfolio is a portfolio with the properties $V_0 = 0$ and $V_1 > 0$. We interpret the existence of an arbitrage portfolio as equivalent to a serious instance of mispricing on the market. In a perfect market, such chances will be wiped out immediately for investors prefer more money to less.

Theorem 5.2: *The market* $\mathrm{BM}(S_0, u, d, p, r)$ *is free of arbitrage if and only if the following conditions hold:*

$$d < e^r < u \tag{5.2}$$

Remark: The condition (5.2) has an easy economic interpretation. It simply says that the return on the stock is not allowed to dominate the return on the bond and vice versa. Which is illustrated in the following figure, where the point (u, d) indicates the (gross) return of stock



We are so lucky in binomial model, for we can easily judge whether the market is free of arbitrage or not. For other models of financial market, we usually assume that the market is absence of arbitrage, we are not able to find tractable restrictions on market parameters to rule out arbitrage opportunities.

In the $\mathrm{BM}(S_0, u, d, p, r)$ market, condition (5.2) is equivalent to the positivity postulate in §1.2.4. We will always assume that condition (5.2) is true, and thus the market is free of arbitrage. In the coming text, we will prove the FTAP (Theorem 1.3) for one-step binomial model: there is a positive linear pricing function $\wp(\cdot)$ in market $\mathrm{BM}(S_0, u, d, p, r)$ such that $\wp(S_1) = S_0$ and $\wp(1) = e^{-r}$.

5.1.3 Completeness

We go on to study pricing problems for financial derivatives in market $\mathrm{BM}(S_0, u, d, p, r)$.

Definition 5.3: A *claim* is any random variable X defined on $(\Omega, \mathcal{F})$ with finite expectation under real world probability measure P. A *financial derivative* is any claim X of the form $X = f(S_1)$, where the function f is called *contract function*.

For financial derivatives, we interpret a given derivative instrument X as a contract which pays

$$X = f(S_1) = \begin{cases} x_u = X(H) = f(S_0 u) & Z = u \\ x_d = X(L) = f(S_0 d) & Z = d \end{cases}$$

dollar to the holder of the contract at time $t = 1$. A typical example would be a European call option on the stock with strike price K, assume that $S_0 d < K < S_0 u$ for practical interests, we see that

$$X = f(S_1) = (S_1 - K)^+ = \max(S_1 - K, 0) = \begin{cases} S_0 u - K & Z = u \\ 0 & Z = d \end{cases}$$

If $X = f(S_0, S_1)$, because S_0 is a known constant, X is effectively a function of S_1 only. The primitive assets are trivial derivatives, they are usually excluded from the set of financial derivatives.

Definition 5.4: A given claim X is said to be *reachable*, or *attainable*, or *marketable*, if there exists a portfolio $\mathbf{h} = [a; b]$ such that

$$V_1 = aS_1 + bB_1 = X$$

In that case we say that the portfolio $\mathbf{h} = [a; b]$ is a *replicating portfolio* of claim X. If all claims can be replicated we say that the market is *complete*.

We see that in a complete market we can in fact replicate all claims, so it is of great interest to investigate when a given market is complete. For the one-step binomial model we find that the market is complete.

Proposition 5.5: The $\mathrm{BM}(S_0, u, d, p, r)$ market is complete.

Proof. For an arbitrary claim X, we want to show that there exists a portfolio $\mathbf{h} = [a; b]$ such that

$$aS_1 + bB_1 = X$$

If we write this out in detail we want to find a solution $[a; b]$ to the following system of equations

$$aS_0u + be^r = x_u \tag{5.3}$$

$$aS_0d + be^r = x_d$$

Since by assumption $u > d$, this linear system has a unique solution, and a simple calculation shows that the replicating portfolio is given by

$$a = \frac{1}{S_0} \cdot \frac{x_u - x_d}{u - d} \tag{5.4}$$

$$b = e^{-r}\frac{x_d u - x_u d}{u - d}$$

□

We see that completeness needs only $u > d$, not the condition (5.2). Thus, in our one-step binomial model, completeness is a property of market that does not require the absence of arbitrage.

Because of the completeness, the payoff of any conceivable asset is in the marketable payoff space X. Let X be a claim, $X(H) = x_u < \infty$ and $X(L) = x_d < \infty$, define

$$\mathbf{x} = \begin{bmatrix} x_u \\ x_d \end{bmatrix}$$

We find that vectors in $\mathbb{R}^2$ and payoffs have one-to-one correspondence, mathematically, the plane $\mathbb{R}^2$ is isomorphic[1] to the marketable payoff space X. As a consequence, a portfolio in the market is equivalent to a linear combination in $\mathbb{R}^2$. For this reason, in binomial model, linear algebra and probability theory can be used interchangeably at one's convenience. The analysis of a claim X can be done by the analysis of the corresponding vector $\mathbf{x}$.

For financial derivatives, the contract function $X = f(S_1)$ can be a nonlinear function. For example, $X = S_1^2$, Eq (5.4) translate the nonlinear function to a linear function of S_1, by the replication portfolio

[1]Isomorphic objects may be considered the same as long as one considers only these properties and their consequences. It is valuable to distinguish between equality and isomorphism: Equality is when two objects are exactly the same, and everything that's true about one object is true about the other, while an isomorphism implies everything that's true about a designated part of one object's structure is true about the other's.

$[a;b]$

$$X = f(S_1) = S_1^2 = aS_1 + bB_1$$

Our main problem is now to determine the "fair" price for a given claim X with contract function f. If such an object exists at all, what is the price X_0 at $t = 0$?

Proposition 5.6: In $\mathrm{BM}(S_0, u, d, p, r)$ market, for any claim X, there is a unique replicating portfolio $\mathbf{h} = [a;b]$ by Equation (5.4), such that $X = aS_1 + bB_1$, and the price of the claim X at $t = 0$, will equal to the price of replicating portfolio

$$X_0 = aS_0 + bB_0$$

Besides, define

$$q = \frac{e^r - d}{u - d} \tag{5.5}$$

We have

$$X_0 = \wp(X) = e^{-r}(qx_u + (1-q)x_d) \tag{5.6}$$

Proof. The existence and uniqueness of $\mathbf{h}$ follows from the fact that the linear system (5.3) has a unique solution. The price of the claim X is

$$X_0 = \wp(X) = \wp(aS_1 + bB_1) = a\wp(S_1) + b\wp(B_1) = aS_0 + b$$

For any $X \in \mathsf{X}$, with q defined in Eq (5.5)

$$e^{-r}(qx_u + (1-q)x_d) = aS_0 + b = X_0$$ □

The q defined in Eq (5.5) has played a significance role in the pricing of derivatives, it deserves an in-depth discussion in the coming section §5.2.

Example 5.1.2: Given market $\mathrm{BM}(S_0, u, d, p, r)$ by

$$S_0 = 4 \quad u = 2 \quad d = 1/2 \quad r = \ln(4/3)$$

Find the price of the European call and put option on the stock with strike price $K = 5$.

By Eq (5.5)

$$q = \frac{e^r - d}{u - d} = \frac{\frac{4}{3} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{5}{9}$$

For the call option with $K = 5$, $x_u = 2 \cdot 4 - 5 = 3$, $x_d = 0$

$$V_0 = e^{-r}(qx_u + (1-q)x_d) = \frac{3}{4}\left(\frac{5}{9} \cdot 3 + 0\right) = \frac{5}{4}$$

For the put option with $K = 5$, $x_u = 0$, $x_d = 5 - \frac{1}{2} \cdot 4 = 3$

$$V_0 = e^{-r}(qx_u + (1-q)x_d) = \frac{3}{4}\left(0 + \left(1 - \frac{5}{9}\right) \cdot 3\right) = 1$$

Using formula (5.6), we do not need to solve linear system (5.3) repeatedly.

Delta hedging: In Equation (5.4), a is the ratio of the change in the derivative price to the change in the stock price as we move between the nodes. We can construct a portfolio of risky assets, the stock S_t and claim X, to produce a fixed return, the risk-free asset. We see that

$$X - aS_1 = bB_1$$

thus if we long one claim and short a shares of stock, we produce a risk-free asset bB_1 in the future time. The construction of a riskless portfolio is sometimes referred to as delta hedging. And the *delta* of the derivative is

$$\frac{x_u - x_d}{S_0 u - S_0 d} = a$$

the ratio of the change in the price of the derivative to the change in the price of the underlying stock.

§ 5.2 Risk Neutral Valuation

When the market is free of arbitrage, the condition (5.2) is true in $\mathrm{BM}(S_0, u, d, p, r)$ market, thus the q in Equation (5.5) must be

$$0 < q < 1$$

By the definition of probability measure (Definition 2.2), we see that the weights in formula (5.6), the q and $1-q$ can be interpreted as probabilities for a new probability measure $\mathbb{Q}$ defined by

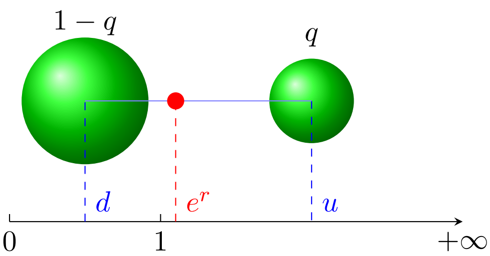
$$\mathbb{Q}(H) = \mathbb{Q}(Z = u) = q$$
$$\mathbb{Q}(L) = \mathbb{Q}(Z = d) = 1 - q$$

Denoting expectation under $\mathbb{Q}$ measure by $\mathrm{E}^{Q}(\cdot)$, we have in $\mathbb{Q}$ world

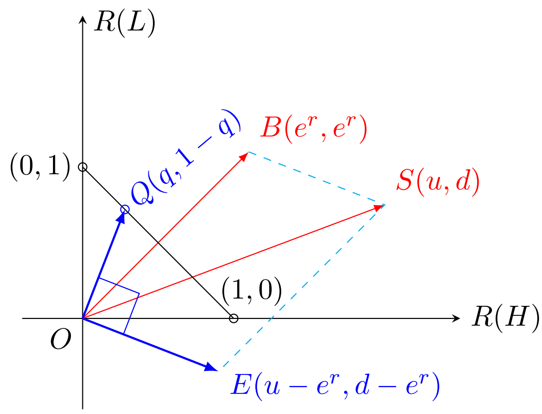
$$\mathrm{E}^{Q}(Z) = qu + (1-q)d = e^r \tag{5.7}$$

Thus, random source Z grows at risk-free rate in $\mathbb{Q}$ world.

From Eq (5.7), we see that if masses q and $1-q$ are attached at the points with coordinates u and d on the real axis, then the centre of mass will be at e^r.



In this barycentric interpretation, the q is a mass. We may also have a geometric interpretation for q



The returns of assets are depicted in $\mathbb{R}^2$: point B is the (gross) return of risk-free asset, point S is the return of stock. $E = S - B$ is the excess return of stock over bond, point Q, located at $(q, 1-q)$, is on the line joining the points (1, 0) and (0, 1). We see that $OQ \perp OE$ since Eq (5.7) gives the orthogonal condition $q(u - e^r) + (1-q)(d - e^r) = 0$. Note that all returns are on the line joining point B and point S, hence point Q is not a return (a return is a payoff with unit price).

5.2.1 Pricing Probability

Considering the following two investments at $t = 0$

(a) Invest S_0 dollars simply at the risk-free asset, the amount at $t = 1$ is $S_0 e^r$

(b) Buy one share of stock at S_0, the expected value at $t = 1$ is $\mathrm{E}^Q(S_1)$ in Q world

Then

$$\mathrm{E}^Q(S_1) = \mathrm{E}^Q(S_0 Z) = S_0 \,\mathrm{E}^Q(Z) = S_0 e^r \tag{5.8}$$

Which exactly states that using probability measure Q, these two investments are equally attractive to risk-neutral investors.

In general, for any payoff X, by formula (5.6)

$$\mathrm{E}^Q(X) = q x_u + (1-q) x_d = X_0 e^r \tag{5.9}$$

Which shows that, under probability measure Q, $\mathrm{E}^Q(X/X_0) = e^r$, the expected return[2] equals the risk-free return. Whatever it is risk-free or risky, whatever it is a derivative or a primary asset, investors require no compensation for risk, the expected return is the risk-free return. In a risk-neutral world all individuals are indifferent to risk, thus a deterministic dollar is equivalent to an expected dollar. For this reason, the world with probability measure Q is risk neutral in this situation, probability measure Q is called *risk neutral probability measure*, Q world is a *risk-neutral world*.

Remark: S_0 and X_0 are numbers. We know that $X_0 = \wp(X)$, but the number X_0 is not necessary bound to the claim X. We can buy X_0 shares of stock, or X_0 dollars of stock (X_0/S_0 shares of stock).

A: Risk Neutral Probability is Just a Tool

Q world is an imaginary world, a mathematical sleight of hand just to simplify derivative pricing. Other than the interpretation of a probability, q is a shortcut to solve the linear system (5.3): We multiply x_u by a number q (to be determined) and x_d by $1-q$, then add them up

$$q x_u + (1-q) x_d = a S_0 [qu + (1-q)d] + b e^r$$

Let's set $qu + (1-q)d = e^r$, then

$$q x_u + (1-q) x_d = a S_0 e^r + b e^r = (a S_0 + b) e^r = X_0 e^r$$

Besides, we find that $q = \frac{e^r - d}{u - d}$ depends only on the market setting, not on the payoff X of given claim. In Q world,

$$q x_u + (1-q) x_d = \mathrm{E}^Q(X)$$

is the expected payoff from the derivative. Thus, the value of the derivative today $X_0 = e^{-r}\,\mathrm{E}^Q(X)$, is its *expected future value discounted at the risk-free rate* (when interest rates are not random).

[2] The log expected return in Q world is

$$\ln(\mathrm{E}^Q(S_1/S_0)) = r$$

However, the expected log return is $\mathrm{E}^Q(\ln(S_1/S_0)) < \ln(\mathrm{E}^Q(S_1/S_0)) = r$. (Jensen's inequality)

Remark: like an auxiliary line in geometry problem, we employ q (determined by the market) to find the price of any claim directly. Using matrix notation, let

$$\mathbf{X} = \begin{bmatrix} S_0 u & S_0 d \\ e^r & e^r \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} S_0 \\ B_0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_u \\ x_d \end{bmatrix}$$

by $\mathbf{X}'\mathbf{h} = \mathbf{x}$, there is unique solution $\mathbf{h} = [a; b] = (\mathbf{X}^{-1})'\mathbf{x}$. Define

$$\mathbf{q} = e^r \cdot \mathbf{X}^{-1}\mathbf{p} = \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

we have formula (5.6)

$$X_0 = aS_0 + bB_0 = \mathbf{h}'\mathbf{p} = \mathbf{x}'\mathbf{X}^{-1}\mathbf{p} = e^{-r}\mathbf{x}'\mathbf{q} = e^{-r}(qx_u + (1-q)x_d)$$

for each payoff $\mathbf{x}$, we do not firstly solve its replicating portfolio $\mathbf{h}$, but employ the market-determined vector $\mathbf{q}$ to compute the price directly.

B: How to Remember the Pricing Probability

In Q world, by formula (5.6)

$$e^r = \mathrm{E}^Q(Z) = qu + (1-q)d$$

solve this equation of q. Setting the probability of the up movement equal to q is, therefore, equivalent to assuming that the expected growth on the stock equals the risk-free interest rate.

5.2.2 Pricing Formula

As we know, it is natural to interpret the variable q in the pricing formula as the probability of an up movement in the stock price.

Theorem 5.7 (Risk Neutral Valuation): *The market* $\mathrm{BM}(S_0, u, d, p, r)$ *is free of arbitrage if and only if there is a risk-neutral probability measure Q, and for any claim (payoff) X, the price is*

$$X_0 = e^{-r}\,\mathrm{E}^Q(X)$$

It is valid to assume (with complete impunity) the world is risk neutral when pricing derivatives. The resulting derivative prices are correct not just in a risk-neutral world, but in the real world as well.

A: Irrelevance of the Real World Probability

We do not make any assumption about the probabilities of up and down movements in order to compute the derivative price. The pricing formula

$$X_0 = \wp(X) = e^{-r}(qx_u + (1-q)x_d)$$

does not involve the probabilities of the stock price moving up or down. For example, we get the same derivative price when the probability of an upward movement is 0.5 as we do when it is 0.9. The price of a derivative is irrelevant to the real world probability, thus irrelevant to the stock's expected return (when r, u and d are fixed). This is surprising and seems counterintuitive.

The key reason is that we are not valuing the derivative in absolute terms. We want to price the derivative in a way that is consistent with the underlying prices given by the market. The idea instead is replication:

- We find the replicating portfolio by solving a system of equations in (5.3), there are no probabilities in this system.
- The replicating portfolio must work on all stock price paths. This replication works regardless of whether the stock goes up or down. They are equal state by state, the probability of each state does not matter.

We create the derivative from trading assets, and determine the price of the derivative in terms of the market prices of the underlying assets. The probabilities of the up and down moves in real world are irrelevant. What matters is the size of the two possible moves (the values of u and d).

Remark: random variables $X = Y$ is irrelevance of probability measure, and replication means $X = Y$ state by state, thus $\wp(X) = \wp(Y)$ has nothing to do with probability measure.

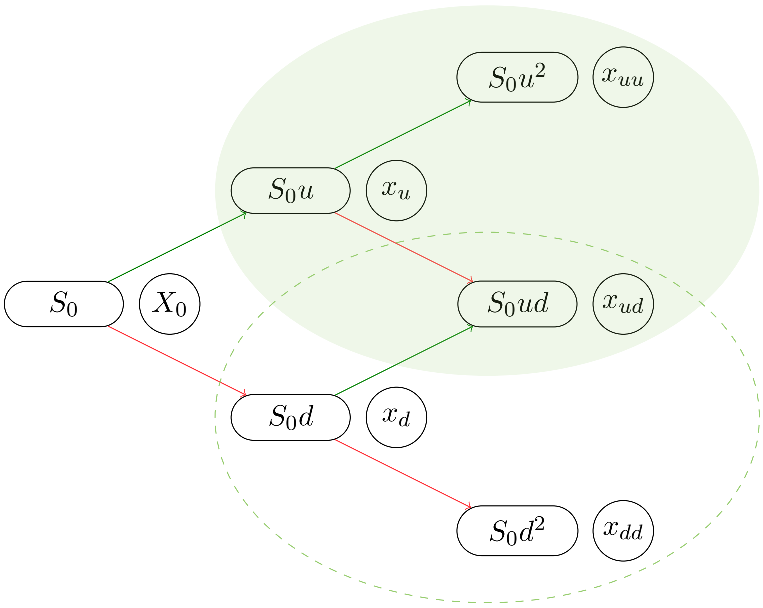
B: Redundance of Derivatives

A financial derivative is defined in terms of some underlying asset which already exists on the market. In a complete market, derivatives can be replaced by underlying asset: Like the binomial model above, there is indeed a unique price for any claim. The price is given by the value of the replicating portfolio, and a negative way of expressing this is as follows: There exists a theoretical price for the claim precisely because of the fact that, strictly speaking, the claim is superfluous—it can equally well be replaced by its replicating portfolio.

Remark: the net supply of any derivative is zero.

§ 5.3 Multi-step Binomial Model

A multi-step binomial model is a stack of one-step binomial models. The following two-step binomial model is a quick illustration: At time $t = 1$, one-step binomial models are repeated for the two states respectively



Let the payoff of a derivative at time 2 is $X = f(S_2)$, with $X(HH) = x_{uu}$, $X(HL) = x_{ud} = X(LH) = x_{du}$ and $X(LL) = x_{dd}$. At time $t = 1$, if there is an up movement, from equation (5.6) we have

$$x_u = e^{-r}(qx_{uu} + (1-q)x_{ud})$$

Repeated application of equation (5.6) for a down movement gives

$$x_d = e^{-r}(qx_{du} + (1-q)x_{dd})$$

Thus, at time $t = 0$

$$\begin{aligned} X_0 &= e^{-r}(qx_u + (1-q)x_d) \\ &= e^{-2r}(q^2x_{uu} + 2q(1-q)x_{ud} + (1-q)^2x_{dd}) = e^{-2r}\,\mathrm{E}^Q(X) \end{aligned}$$

Not surprisingly, we arrive at a risk neutral pricing formula since q can be interpreted as the probability of an up movement. However, note that there are intermediate tradings, we rebalance the replication portfolio (Eq 5.4) at $t = 1$.

5.3.1 Market Setting

We can generalize the model to T time steps: As before we have two underlying assets, a risk-free bond with price process B_t and a stock with price process S_t. We assume a constant deterministic period rate of interest at r (continuous compounding). This means that the bond price dynamics are

$$B_t = e^{rt} \qquad t = 0, 1, 2, \cdots, T$$

Let $Z_1, Z_2, \cdots, Z_T$ be i.i.d up-and-down random variables, taking only the two values u and d with

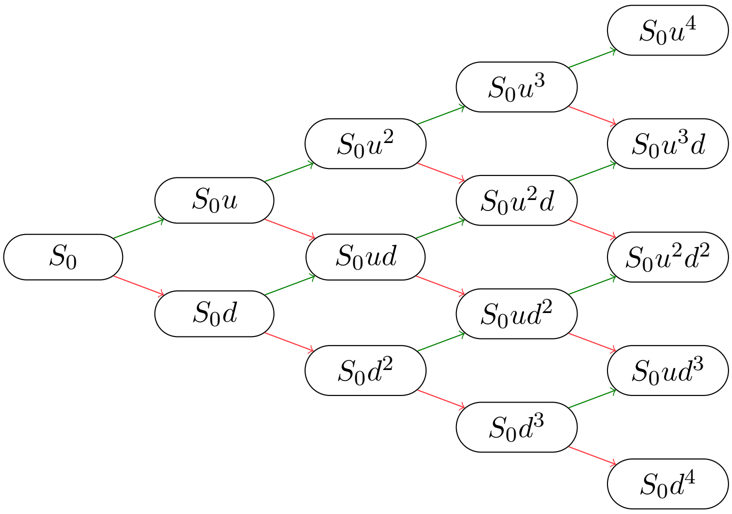
$$\mathrm{P}(Z_t = u) = p \qquad \mathrm{P}(Z_t = d) = 1 - p \qquad t = 1, 2, \cdots, T$$

The dynamics of the stock price are given by the following stochastic process

$$S_t = S_{t-1} Z_t \qquad t = 1, 2, \cdots, T$$

We can illustrate the stock dynamics by means of a tree, as in Figure 5.2. Note that the tree recombines in the sense that an up movement followed by a down movement leads to the same stock price as a down movement followed by an up movement.

Figure 5.2: Multi-step Binomial Model There are two assets: the bond price dynamics are $B_t = e^{rt}$, and the stochastic process for stock price are $S_t = S_{t-1} Z_t$, where the i.i.d up-and-down random variables $Z_1, Z_2, \cdots, Z_T$ are the risk source of the market. The tree recombines, every non-leaf node starts a one-step binomial model. An example of $T = 4$ gives the following tree



We assume that today's stock price S_0 is known at time 0, as are the positive constants u, d, and p. What is more, we assume that condition (5.2) is true, say, $d < e^r < u$. For brevity, the market setting of T-step binomial model will be denoted by $\mathrm{BM}_T(S_0, u, d, p, r)$.

In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, we assume that the market is driven by a sequence of coin tosses, we define the sample space naturally as

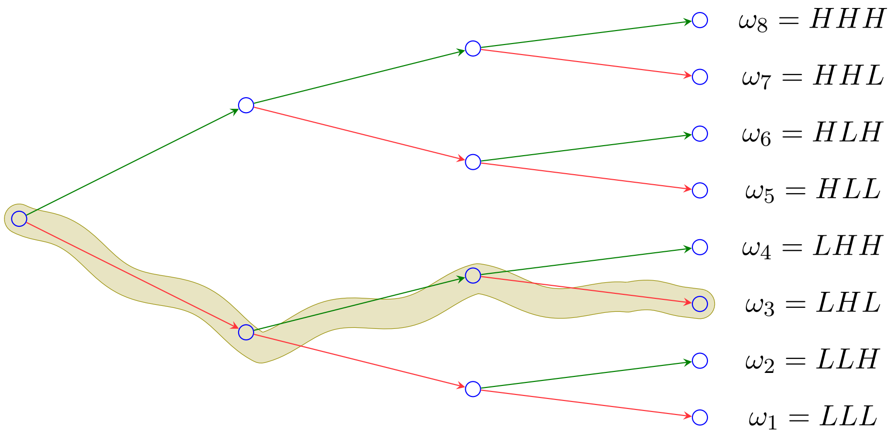
$$\Omega = \{w_1 w_2 \cdots w_T : w_t \in \{H, L\}, t = 1, 2, \cdots, T\}$$

and the event space as the power set of sample space, $\mathcal{F} = 2^\Omega$. Let $\mathbb{I}_t$ be the information available up to time t, then $\mathbb{I}_0 = \{\emptyset, \Omega\}$, and (since $Z_t = S_t / S_{t-1}$)

$$\mathbb{I}_t = \sigma(Z_1, Z_2, \cdots, Z_t) = \sigma(S_1, S_2, \cdots, S_t) \qquad t = 1, 2, \cdots, T$$

As time goes by, the information set becomes larger, $\mathbb{I}_{t-1} < \mathbb{I}_t$. Note that $\mathbb{I}_t \neq \sigma(S_t)$ and $\mathbb{I}_T = \mathcal{F}$. A sequence of coin tosses $w_1 w_2 \cdots w_T$ is called a *state* or *path*, there are 2^T paths. For each path $w_1 w_2 \cdots w_T$ there is a unique number $i = 1 + \sum_{t=1}^{T} 2^{T-t} 1_{w_t = H}$, we say $w_1 w_2 \cdots w_T$ is the i-th state or path, and denote it by ω_i. An example with $T = 3$ is shown in Figure 5.3, where $\omega_3 = LHL$ marks the path of down-up-down.

Figure 5.3: States of World Tossing a coin three times, a sequence of coin tosses $w_1w_2w_3$ is a path. We can number the path $w_1w_2w_3$ by the number $i = 1 + \mathit{1}_{w_1=H} + 2\cdot \mathit{1}_{w_2=H} + 4\cdot \mathit{1}_{w_3=H}$, and denote it by ω_i. Clearly, the number of path LHL is 3, hence $\omega_3 = LHL$. The collection of paths $\{\omega_1, \omega_2, \cdots, \omega_8\}$ is the sample space Ω.



Remark: In a probability world, the sample space is known, a doable event space is selected and a probability measure is equipped with. A market modelled by probability theory is limited to a world of *deterministic uncertainty* or *known unknowns*. The real world is uncertain, there are *unknown unknowns*, we may not know the full states of future world, even we know the sample space, we may not know the real world probability. A world govern by probability model is deterministic in the sense that we know the deterministic set of outcomes and probabilities for sure, the uncertainty part is which outcome will be realized.

5.3.2 Self-financing Process

There are only two primary assets in the multi-step binomial model, we have the following definition of self-financing portfolio as a special case of Definition 1.2.

Definition 5.8: In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, the rebalancing of portfolio is *self-financing* if

$$\mathbf{h}_t = [a_t; b_t] \in \mathbb{I}_t \qquad t = 0, 1, 2, \cdots, T-1$$

such that $\mathbf{h}_t$ is a function of $S_0, S_1, \cdots, S_t$, and

$$a_{t-1}S_t + b_{t-1}B_t = a_tS_t + b_tB_t \qquad t = 1, 2, \cdots, T-1 \tag{5.10}$$

A portfolio process $\{\mathbf{h}_t\}$ is *self-financing* if each rebalancing is self-financing. Let $\mathbf{h}_T = [a_{T-1}; b_{T-1}]$, the holding sequence $\{\mathbf{h}_t\}_{t=0}^{T}$ of a self-financing portfolio is called a *self-financing trading strategy* or *portfolio strategy*. And the set of all portfolio strategies is denoted by $\mathbb{S}$.

Remark: For clarity, a portfolio process $\{\mathbf{h}_t\}$ will be simply written by portfolio $\mathbf{h}_t$ when it is clear that we are not mentioning the holding at time t but the dynamics of the portfolio.

We allow the holdings to be a *predictable* contingent strategy, i.e. the portfolio we buy at time t is allowed to depend on all information up to time t, by observing the evolution of the stock price. We are, however, not allowed to look into the future. Thus, all portfolio rebalancings are predictable unless explicitly stated. The value process corresponding to the portfolio $\mathbf{h}_t$ is defined by

$$V_t = V(\mathbf{h}_t) = a_t S_t + b_t B_t \qquad t = 0, 1, 2, \cdots, T$$

Remark: At time T, $V_T = a_T S_T + b_T B_T$, we set $a_T = a_{T-1}$ and $b_T = b_{T-1}$ without rebalancing.

At time t, once the asset prices, B_t and S_t are revealed to the investor (price is right continuous), we change the holdings from $\mathbf{h}_{t-1}$ to $\mathbf{h}_t$ in response to the arrival of the new information. We hold portfolio $\mathbf{h}_t = [a_t; b_t]$ at period $t+1$ for the time interval $(t, t+1]$. The entity V_t above is of course the market value of the portfolio $[a_t; b_t]$ at time t. More exactly, V_t gives the portfolio value at the moment right after the portfolio is rebalanced due to the new price.

Example 5.3.1 (Understanding the Self-financing Condition): The stock prices of IBM and MS with a portfolio rebalancing are as follows

	Price		Portfolio	
	IBM	MS	IBM	MS
$t=0$	10	12	8	11
$t=1$	15	9	5	16

At $t=1$, we see that the old (portfolio's) value equals new value (sell old buy new)

$$8 \cdot 15 + 11 \cdot 9 = 219 = V_1 = 5 \cdot 15 + 16 \cdot 9$$

Since we sell 3 ($5 - 8 = -3$) shares of IBM, and use the proceeds to buy 5 ($16 - 11 = 5$) shares of MS, $3 \cdot 15 = 45 = 5 \cdot 9$, this rebalancing is self-financing. The change of portfolio value is $\Delta V_1 = V_1 - V_0 = 219 - (8 \cdot 10 + 11 \cdot 12) = 7$, which is caused by the change of assets' price

$$\Delta V_1 = 7 = 8 \cdot (15 - 10) + 11 \cdot (9 - 12)$$

Observe that at $t = 1$

$$\begin{aligned} 219 &= 2 \cdot 15 + 21 \cdot 9 = 8 \cdot 15 + 11 \cdot 9 \\ &= 11 \cdot 15 + 6 \cdot 9 = 14 \cdot 15 + 1 \cdot 9 \end{aligned}$$

surely we can form may portfolios having the same value as V_1. Thus, these rebalancings are self-financing, and there are various possibilities for self-financing trading strategies.

Remark: A self-financing portfolio is a portfolio without any exogenous infusion or withdrawal of money at each rebalancing (self-financing condition 5.10). A self-financing trading strategy is characterized by $V_{t+0} = V_t$, the values of the portfolio just before and after any transactions are equal, for the accession of a new asset has to be financed through the sale of some other asset.

- Let $\Delta a_t = a_{t+1} - a_t$, and $\Delta b_t = b_{t+1} - b_t$, then
$$B_t \Delta b_{t-1} = -S_t \Delta a_{t-1} \qquad \forall t$$
Which states that increasing number of bond is done by selling stocks —— decreasing number of stock, and vise versa.
- Transactions do not change the value of portfolio
$$V_{t+0} - V_t = S_t \Delta a_{t-1} + B_t \Delta b_{t-1} = 0$$
The rebalancing of portfolio do not change the value of the portfolio
- Gain of value process: For
$$\Delta V_t = V_{t+1} - V_t = (a_t S_{t+1} + b_t B_{t+1}) - (a_t S_t + b_t B_t) = a_t \Delta S_t + b_t \Delta B_t$$
the value change is caused by the change of assets' prices, not by the change of the holdings

5.3.3 Derivative Pricing: P World

Derivatives are trading in real world, however, it is more convenient to pricing in Q world. The binomial algorithm is the key of binomial model, and the linkage between the P world and Q world.

A: Reachable Claim

For financial derivatives, we have a contract function, such that X is a function of price or price process of some underlying assets. Thus, a claim is a random variable $X \in \mathbb{I}_T$ with finite expectation. A European option may be exercised only at the expiration date T. On the other hand, an American option may be exercised at any time prior to and including its maturity date, $t \leqslant T$. I like to call European style derivatives T-claim, for they cease to exist (execute or expire) exactly at maturity time T.

Definition 5.9: A given T-claim X is said to be *reachable*, or *attainable*, or *marketable*, if there exists a self-financing portfolio $\mathbf{h}_t$ such that
$$V_T = V(\mathbf{h}_T) = X$$
In that case we say that the portfolio $\mathbf{h}_t$ is a *replicating portfolio* of claim X.

The interpretation of T-claim is that the holder of the contract receives the random amount X at time T. We are most considering claims that are "simple", in the sense that the payoff of the claim only depends on the value S_T of the stock price at the final time T.

Definition 5.10: A T-claim X is *simple* if
$$X = f(S_T) \in \sigma(S_T)$$
where the contract function $f(\cdot)$ is some given real valued function.

It is possible to consider derivatives which depend on the entire path of the price process during the interval $[0, T]$
$$X = f(S_1, S_2, \cdots, S_T) \in \mathbb{I}_T$$

but then the theory for path dependent derivatives becomes a little more complicated. We will investigate its algorithm a bit later.

B: Binomial Algorithm

It is clear from the construction that the price process of stock at time t can be written as

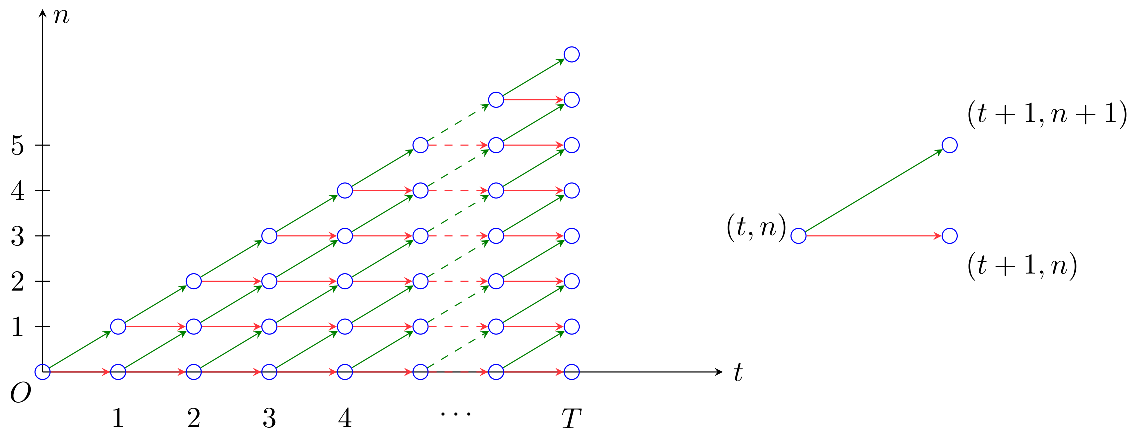
$$S_{t,n} = S_0 u^n d^{t-n} \qquad t = 1, 2, \cdots, T \quad n = 0, 1, 2, \cdots, t$$

where n denotes the number of up-moves that have occurred. Thus each node in the binomial tree can be represented by a pair (t, n) with $n = 0, 1, 2, \cdots, t$. And

$$S_{t+1,n} = S_{t,n} d \qquad S_{t+1,n+1} = S_{t,n} u$$

Note that the stock price does not follows binomial distribution, but the number of upward price movements, n, follows binomial distribution.

Figure 5.4: Binomial Algorithm In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, at each non-leaf node (t, n), there is a one-step binomial model. Thus, we can work backward to find the price of a simple T-claim.



A multi-step binomial tree is depicted in Figure 5.4, by repeating one-step binomial model, we have the following algorithm.

Algorithm 5.11 (Binomial Algorithm): In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, any simple T-claim $X = X_T = f(S_T)$ can be replicated using a self-financing portfolio. If $X_{t,n}$ denotes the value of the portfolio at the node (t, n), then at each final node

$$X_{T,n} = f(S_0 u^n d^{T-n})$$

and at earlier nodes, $X_{t,n}$ can be computed recursively by the scheme

$$X_{t,n} = e^{-r}(qX_{t+1,n+1} + (1-q)X_{t+1,n}) \tag{5.11}$$

Where $q = \frac{e^r - d}{u-d}$. The self-financing replicating portfolio is given by

$$
\begin{aligned}
a_{t,n} &= \frac{X_{t+1,n+1} - X_{t+1,n}}{S_{t+1,n+1} - S_{t+1,n}} = \frac{X_{t+1,n+1} - X_{t+1,n}}{(u-d)S_{t,n}} \\
b_{t,n} &= \frac{X_{t+1,n} - a_{t,n}S_{t+1,n}}{B_{t+1}} = \frac{X_{t+1,n}u - X_{t+1,n+1}d}{(u-d)B_{t+1}}
\end{aligned}
\tag{5.12}
$$

In particular, the arbitrage free price of the claim at $t = 0$ is given by $X_0 = X_{0,0}$.

Proof. Each non-leaf node (t, n) starts an immediate one-step binomial model $\mathrm{BM}(S_{t,n}, u, d, p, r)$, with $x_u = X_{t+1,n+1}$ and $x_d = X_{t+1,n}$. The replicating portfolio $[a_{t,n}; b_{t,n}]$ produces

$$a_{t,n}S_{t+1,n} + b_{t,n}B_{t+1} = X_{t+1,n}$$

$$a_{t,n}S_{t+1,n+1} + b_{t,n}B_{t+1} = X_{t+1,n+1}$$

Similar to Equation (5.4), we have Equation (5.12). By Equation (5.6) from Proposition 5.6

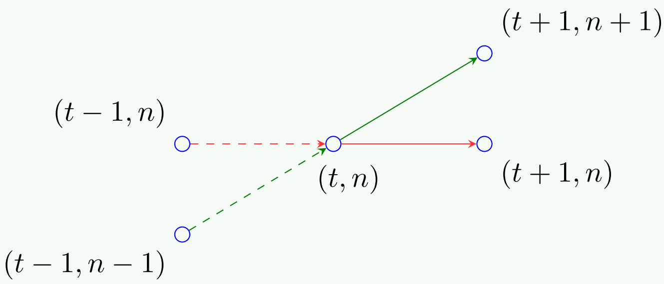
$$X_{t,n} = a_{t,n}S_{t,n} + b_{t,n}B_t = e^{-r}(qX_{t+1,n+1} + (1-q)X_{t+1,n}) \qquad n = 0, 1, 2, \cdots, t$$

Which gives Equation (5.11) for $t = T-1, \cdots, 1, 0$. Note that we are working backward.

Given any t in $\{1, 2, \cdots, T-1\}$, at node (t, n) we see that

$$a_{t,n}S_{t,n} + b_{t,n}B_t = X_{t,n} = a_{t-1,n}S_{t,n} + b_{t-1,n}B_t \qquad n = 0, 1, 2, \cdots, t$$

The first equality reads $X_{t,n}$ as the value of derivative at node (t, n), while the second equality reads $X_{t,n}$ as the payoff at node (t, n) replicated by node $(t-1, n)$ (if $n < t$) and $(t-1, n-1)$ (if $n > 0$):



Thus, at time t

$$a_tS_t + b_tB_t = X_t = a_{t-1}S_t + b_{t-1}B_t$$

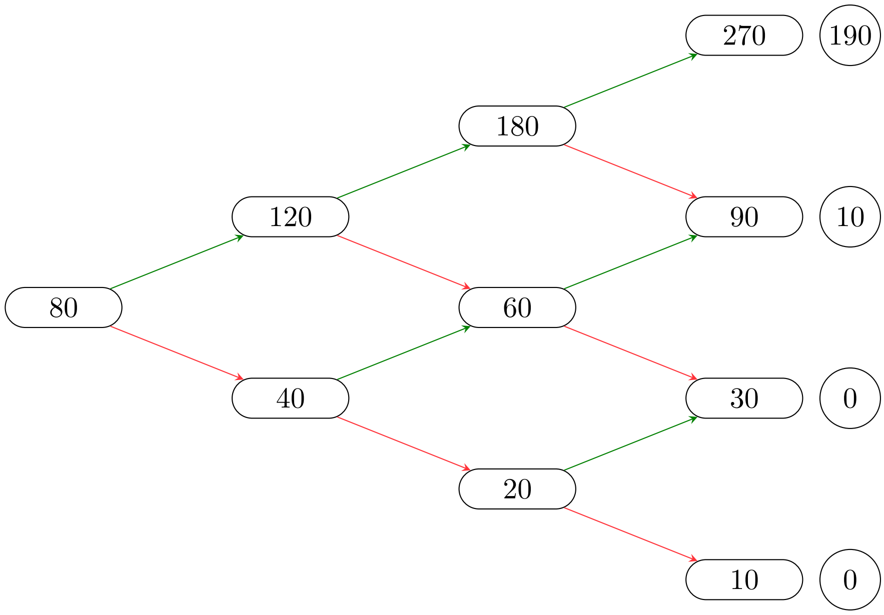
the replicating portfolio is self-financing. □

Remark: Let (a_t, b_t) be the self-financing replicating portfolio at time t, then random variables $a_t, b_t \in \mathbb{I}_t$, and $X = a_{T-1}S_T + b_{T-1}B_T$. Because the portfolio strategy is self-financing, we have

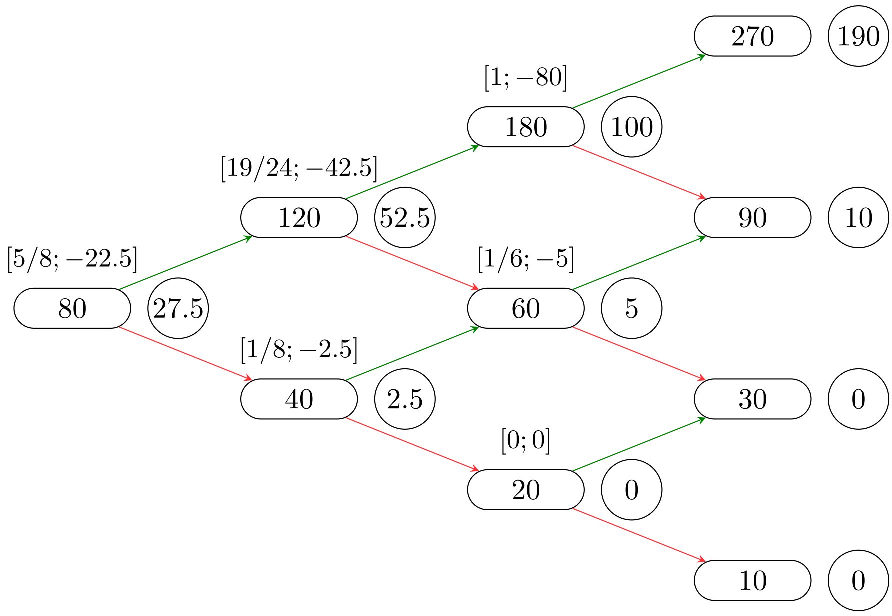
$$a_0S_0 + b_0B_0 = X_0 = \wp(X) = \wp(a_{T-1}S_T + b_{T-1}B_T)$$

Example 5.3.2: In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, let $T = 3$, $S_0 = 80$, $u = 1.5$, $d = 0.5$, $p = 0.6$, and for computational simplicity, $r = 0$. Find the price of a European call $C(S, T, K)$ on the underlying stock with $K = 80$.

Following algorithm 5.11, first we plot the tree and compute the payoff of the call



Then we working backward



We have computed prices of the call at each non-leaf node, along with the dynamic replicating portfolios.

C: No-arbitrage and Completeness

An arbitrage portfolio is defined by condition (1.13), with respect to $\mathrm{BM}_T(S_0, u, d, p, r)$ market, an arbitrage possibility is a self-financing portfolio $[a_t; b_t]$ with value process $V_t = a_t S_t + b_t B_t$ having the following properties

$$V_0 = 0, \; V_T \gneq 0$$

Remark: Why the intermediate step is not mentioned? Cause the portfolio $[a_t; b_t]$ is self-financing.

If there is an arbitrage opportunity at some step $t < T$, then we can liquidate all risky assets and invest all the money in bond at time t and carry forward to end time T.

The one-step binomial model is the building block of multi-step binomial model, the condition for no-arbitrage and completeness of $\mathrm{BM}_T(S_0, u, d, p, r)$ follows straightforward from the results of $\mathrm{BM}(S_0, u, d, p, r)$. If condition (5.2) is true, $d < e^r < u$, the market $\mathrm{BM}_T(S_0, u, d, p, r)$ is free of arbitrage. Conversely, if the market is free of arbitrage, the condition (5.2) must be true.

Proposition 5.12: The market $\mathrm{BM}_T(S_0, u, d, p, r)$ is free of arbitrage if and only if $d < e^r < u$.

If all T-claims (not only *simple* claims) can be replicated we say that the market is complete.

Proposition 5.13: The multi-step binomial model $\mathrm{BM}_T(S_0, u, d, p, r)$ is complete, i.e. every T-claim is attainable (can be replicated by a self-financing portfolio).

Remark: The $\mathrm{BM}_T(S_0, u, d, p, r)$ market is *dynamically complete*, for the replicating portfolio given by Equation (5.12), $\mathbf{h}_t = [a_t; b_t] \in \mathbb{I}_t$, is rebalanced every step. If there is no portfolio rebalancing, the payoff space of static portfolios is only a subset of T-claims. The payoff space of all T-claims is $\mathsf{L}^2(\Omega, \mathcal{F}, \mathrm{P})$ (here $\mathsf{L}^2(\Omega, \mathcal{F}, \mathrm{P}) = \mathsf{L}^1(\Omega, \mathcal{F}, \mathrm{P})$), the attainable payoff space is

$$\mathsf{X} = \{a_T S_T + b_T B_T : \{[a_t; b_t]\}_{t=0}^T \in \mathbb{S}\}$$

where $\mathbb{S}$ contains all self-financing trading strategies. The completeness of $\mathrm{BM}_T(S_0, u, d, p, r)$ shows that $\mathsf{X} = \mathsf{L}^2(\Omega, \mathcal{F}, \mathrm{P})$. Hence, X is isomorphic to $\mathbb{R}^{2^T}$, intermediate tradings can expand the dimension of marketable payoff space.

If the claim is path-dependent, the Equation (5.11) in Algorithm 5.11 is updated by

$$X_t(w_1 w_2 \cdots w_t) = e^{-r}(q X_{t+1}(w_1 w_2 \cdots w_t H) + (1-q) X_{t+1}(w_1 w_2 \cdots w_t L)) \tag{5.13}$$

for any state (path) beginning with $w_1 w_2 \cdots w_t$. Note that we restore the simplified recombining tree to a non-recombining tree as in Figure 5.5.

5.3.4 Derivative Pricing: Q World

If we define probability measure Q by

$$\mathrm{Q}(w_{t+1} = H \,|\, \mathbb{I}_t) = \mathrm{Q}(Z_{t+1} = u \,|\, \mathbb{I}_t) = q \qquad t = 0, 1, \cdots, T-1$$

then we have

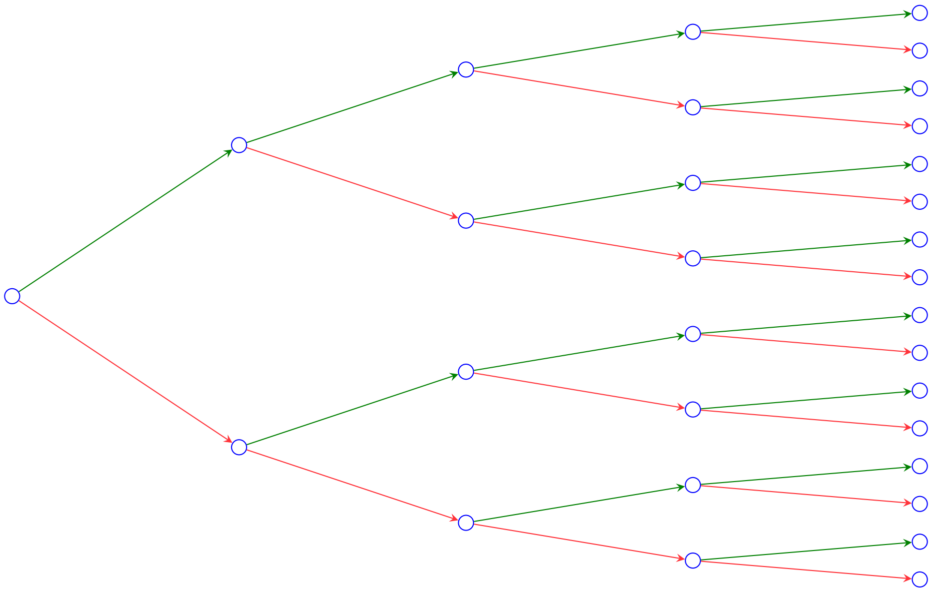
$$\mathrm{Q}(w_{t+1} = L \,|\, \mathbb{I}_t) = \mathrm{Q}(Z_{t+1} = d \,|\, \mathbb{I}_t) = 1 - q$$

Consequently, the probability of state $w_1 w_2 \cdots w_T$ in Q world[3] is

$$\mathrm{Q}(w_1 w_2 \cdots w_T) = q^n (1-q)^{T-n}$$

[3]In Algorithm 5.11, $\mathrm{Q}(w_{t+1} = H \,|\, S_t) = q$ is read effortlessly from the recombining tree. However, it only works for simple T-claim $X \in \sigma(S_T)$, for it will not define a probability measure uniquely, see Exercise 5.11. The recombining tree in Figure 5.4 is only a simplification for computation purpose, it is readily to be restored to a full tree as Figure 5.5 with 2^t nodes at time t. In a full tree, we interpret q plainly as the conditional probability, $\mathrm{Q}(w_{t+1} = H \,|\, \mathbb{I}_t) = q$.

Figure 5.5: Non-recombining Tree In $\mathrm{BM}_T(S_0, u, d, p, r)$ market, if u or d is path dependent, such that $S_t(w_1 \cdots w_{t-2}HL) \neq S_t(w_1 \cdots w_{t-2}LH)$, or if the claim is path-dependent, we need a non-recombining tree. At time t, there are 2^t nodes in a non-recombining tree.



where $n = \sum_t 1_{w_t=H}$ is the number of up-moves. For each single step (coin toss)

$$\mathrm{Q}(w_t = H) = \mathrm{Q}(Z_t = u) = q$$

$$\mathrm{Q}(w_t = L) = \mathrm{Q}(Z_t = d) = 1 - q$$

Comparing with P world

$$\mathrm{P}(w_1 w_2 \cdots w_T) = p^n (1-p)^{T-n}$$

to change from P world to Q world, simply change the probability of the up movement from p to q.

Remark: Probability measure P and Q are equivalent.

- $Z_1, Z_2, \cdots, Z_T$ are still i.i.d in Q world (Exercise 5.12).
- If a self-financing portfolio $\mathbf{h}$ is an arbitrage opportunity in P world, then it is an arbitrage opportunity in Q world, and vice versa.

A: Pricing Formula

In Q world, we have

$$\mathrm{E}_t^Q(Z_{t+1}) = \mathrm{E}^Q(Z_{t+1}) = e^r \qquad t = 0, 1, 2, \cdots, T-1$$

and

$$\mathrm{E}_t^Q(S_{t+1}) = \mathrm{E}^Q\left(S_{t+1} \,|\, \mathbb{I}_t\right) = S_t e^r \qquad t = 0, 1, 2, \cdots, T-1$$

Furthermore, if V_t is the value of a self-financing process, then

$$\mathrm{E}_t^Q(V_{t+1}) = e^r V_t \tag{5.14}$$

Which is quite similar to Eq (5.9), thus probability measure Q is called *risk neutral probability measure*, Q world is a *risk-neutral world*.

In Q world, Equation (5.13) amounts to

$$X_t = e^{-r}\, \mathrm{E}_t^Q(X_{t+1}) \qquad t = 0, 1, \cdots, T-1 \tag{5.15}$$

For X_t is the value process of a self-financing replicating portfolio. As a consequence, for any T-claim, either simple or path-dependent T-claim, we have the following pricing formula.

Theorem 5.14: *The arbitrage free price at t of a T-claim X is given by*

$$X_t = e^{-r(T-t)}\, \mathrm{E}_t^Q(X)$$

where Q *denotes the risk neutral probability measure. In particular, at $t = 0$*

$$X_0 = e^{-rT}\, \mathrm{E}^Q(X) = e^{-rT} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} f\left(S_0 u^n d^{T-n}\right)$$

Proof. The market is complete, X can be replicated by a self-financing portfolio. By Eq (5.15)

$$\begin{aligned} X_t &= e^{-r}\, \mathrm{E}_t^Q(X_{t+1}) = e^{-r}\, \mathrm{E}_t^Q(e^{-r}\, \mathrm{E}_{t+1}^Q(X_{t+2})) \\ &= e^{-2r}\, \mathrm{E}_t^Q(X_{t+2}) = \cdots = e^{-r(T-t)}\, \mathrm{E}_t^Q(X) \end{aligned}$$

Let Y denote the number of up-moves in the tree

$$Y = \sum_{t=1}^{T} 1\left(Z_t = u\right)$$

then Y has a binomial distribution in Q world, $Y \sim \mathrm{B}^Q(T, q)$, and

$$X = f\left(S_T\right) = f\left(S_0 u^Y d^{T-Y}\right)$$

Thus

$$X_0 = e^{-rT}\, \mathrm{E}^Q(X) = e^{-rT} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} f\left(S_0 u^n d^{T-n}\right) \qquad \square$$

B: Martingale Representation

Let's normalized the asset prices by the price of risk-free bond B_t

$$S_{:t} = S_t / B_t \qquad B_{:t} = B_t / B_t = 1$$

We say that B_t is the *numéraire*, the asset in which values of other assets are measured. For convenience, this price system is called B *price system*. The price $S_{:t}$ is sometimes called discounted stock price, for $1/B_t = e^{-rt}$ is a discounted factor (when the risk-free rate is deterministic).

Proposition 5.15: A portfolio $\mathbf{h}_t = [a_t; b_t]$ is self-financing if and only if $\Delta V_{:t} = a_t \Delta S_{:t}$.

Proof. For

$$\begin{aligned}\Delta V_{:t} - a_t \Delta S_{:t} &= a_{t+1} S_{:t+1} + b_{t+1} - (a_t S_{:t} + b_t) - a_t(S_{:t+1} - S_{:t}) \\ &= a_{t+1} S_{:t+1} + b_{t+1} - (a_t S_{:t+1} + b_t) \\ &= [a_{t+1} S_{t+1} + b_{t+1} B_{t+1} - (a_t S_{t+1} + b_t B_{t+1})]/B_{t+1}\end{aligned}$$

thus condition (5.10) holds if and only if $\Delta V_{:t} = a_t \Delta S_{:t}$. □

In B price system, $S_{:t}$ is a Q martingale. What's more, any value process $V_{:t}$ of a self-financing portfolio is a Q martingale.

Proposition 5.16: In B price system, the value process $V_{:t}$ of a self-financing portfolio is a martingale under measure Q

$$\mathrm{E}^Q(V_{:t+1} \mid \mathbb{I}_t) = V_{:t}$$

The following theorem asserts that the converse is also true: there is a replicating portfolio for each Q martingale.

Theorem 5.17 (Binomial Representation Theorem): *Given measure Q such that $S_{:t}$ is a martingale, for any Q martingale $V_{:t}$, there exist a unique predictable process a_t such that*

$$V_{:t} = V_{:0} + \sum_{u=1}^{t} a_{u-1}(S_{:u} - S_{:u-1}) \tag{5.16}$$

Proof. At time t, the state $w_1 w_2 \cdots w_t$ is known. For clarity, we suppress $w_1 w_2 \cdots w_t$ and write $V_{:t+1}(w_1 w_2 \cdots w_t w_{t+1})$ as $V_{:t+1}(w_{t+1})$. Given $V_{:t}$ and $S_{:t}$, $V_{:t+1}$ and $S_{:t+1}$ can take on one of two possible values that we denote by $\{V_{:t+1}(H), V_{:t+1}(L)\}$ and $\{S_{:t+1}(H), S_{:t+1}(L)\}$ respectively. Since $V_{:t}$ and $S_{:t}$ are Q martingale ($q = \mathrm{Q}(w_{t+1} = H \mid \mathbb{I}_t)$)

$$\begin{aligned}\mathrm{E}_t^Q(V_{:t+1}) &= qV_{:t+1}(H) + (1-q)V_{:t+1}(L) = V_{:t} \\ \mathrm{E}_t^Q(S_{:t+1}) &= qS_{:t+1}(H) + (1-q)S_{:t+1}(L) = S_{:t}\end{aligned}$$

Solving both equations for q leads to the relation

$$q = \frac{V_{:t} - V_{:t+1}(L)}{V_{:t+1}(H) - V_{:t+1}(L)} = \frac{S_{:t} - S_{:t+1}(L)}{S_{:t+1}(H) - S_{:t+1}(L)}$$

Which in turn implies that

$$\frac{V_{:t+1}(H) - V_{:t+1}(L)}{S_{:t+1}(H) - S_{:t+1}(L)} = \frac{V_{:t+1}(L) - V_{:t}}{S_{:t+1}(L) - S_{:t}} = \frac{V_{:t+1}(H) - V_{:t}}{S_{:t+1}(H) - S_{:t}} \equiv a_t \tag{5.17}$$

Clearly, a_t is predictable, $a_t \in \mathbb{I}_t$. For $V_{:t+1}(H)$, $V_{:t+1}(L)$, $S_{:t+1}(H)$ and $S_{:t+1}(L)$ are known given $\mathbb{I}_t$ (read the coming remark on $V_{:t+1}(H) \in \mathbb{I}_t$). Equation (5.17) shows that for any w_{t+1}

$$V_{:t+1} - V_{:t} = a_t(S_{:t+1} - S_{:t})$$

and Equation (5.16) is evident.

For uniqueness, if a predictable process A_t satisfies Equation (5.16), then

$$V_{:t+1} - V_{:t} = A_t(S_{:t+1} - S_{:t})$$

Accordingly, given $w_{1:t} = w_1 w_2 \cdots w_t$, for any w_{t+1}

$$A_t(w_{1:t}) = \frac{V_{:t+1}(w_{1:t}w_{t+1}) - V_{:t}(w_{1:t})}{S_{:t+1}(w_{1:t}w_{t+1}) - S_{:t}(w_{1:t})} = a_t(w_{1:t})$$

thus a_t is uniquely determined. $\square$

Remark: $V_{:t+1} \in \mathbb{I}_{t+1}$, however $V_{:t+1}(H) \in \mathbb{I}_t$. For at time t, $w_{1:t} = w_1 w_2 \cdots w_t$ is given, $V_{:t+1}(H) = V_{:t+1}(w_{1:t}H) \in \mathbb{I}_t$ contains no uncertainty. However, at time $t-1$, $V_{:t+1}(H)$ is a random variable: since $w_{t+1} = H$ is given but w_t is to be resolved, $V_{:t+1}(H)$ takes value in $\{V_{:t+1}(w_{1:t-1}HH),$ $V_{:t+1}(w_{1:t-1}LH)\}$. For example, at time 1, $w_2 \in \{H, L\}$ is not resolved yet, if $w_1 = H$, then $V_{:3}(H) \in \{V_{:3}(HHH), V_{:3}(HLH)\}$ is the restriction of $V_{:3}$ to the path such that $w_1 = H$ and $w_3 = H$. Thus, $V_{:3}(H)$ is a random variable at time 1, please find out the corresponding two-step sub-tree in Figure 5.5 for an illustration.

Equation (5.17) is consistent with Equation (5.12), also the delta equals

$$a_t = \frac{\Delta V_{:t}}{\Delta S_{:t}} = \frac{V_{:t+1} - V_{:t}}{S_{:t+1} - S_{:t}} \in \mathbb{I}_t$$

In the time period from t to $t+1$, the replicating portfolio holds a_t shares of stocks and $b_t = V_{:t} - a_t S_{:t}$ shares of risk-free bonds.

In $\mathrm{BM}_T(S_0, u, d, p, r)$ market, if derivative securities are traded, their price processes $V_{:t}$ must be martingales under risk-neutral measure. (For V_T is reachable and $\mathrm{E}^Q(V_{:T} \,|\, \mathbb{I}_t) = V_{:t}$)

to do: using martingale to solve BM (like BS) (1) no arbitrage (one-step then multi-step). (2) eq (2.28) and find $q = \frac{e^r - d}{u-d}$, $\mathrm{Q}(w_{t+1} = H \,|\, \mathbb{I}_t) = q$ and $\mathrm{E}^Q_t(S_{t+1}) = \mathrm{E}^Q\left(S_{t+1} \,|\, \mathbb{I}_t\right) = S_t e^r$ (or working backward, or unconditional probability, or by BM algorithm find state price). (3) pricing function $V_t = e^{-r}\,\mathrm{E}^Q_t(V_{t+1})$. (4) complete, define $X_t = e^{-r}\,\mathrm{E}^Q_t(X_{t+1})$, and BRT.

§ 5.4 Exercise

5.1 Does Z in Equation (5.1) follow Binomial distribution? What are $\mathrm{E}(Z)$ and $\mathrm{var}(Z)$?

5.2 Prove that if condition (5.2) hold, then $\mathrm{BM}(S_0, u, d, p, r)$ is free of arbitrage.

5.3 Show that $V_0(\mathbf{h}) = \wp(V_1(\mathbf{h}))$.

5.4 From Eq (5.4), show that
$$a = \frac{X/B_1 - X_0}{S_1/B_1 - S_0}$$

5.5 Given market $\mathrm{BM}(S_0 = 75, u = 1.2, d = 0.9, p = 1/2, r = \ln(1.1))$. Suppose that you can borrow money at $r_b = \ln(1.12)$, but the rate for deposits is lower at $r_d = \ln(1.08)$. Find the values of your replicating portfolios for a European put and a call with maturity at time 1 for strike price $K = 75$.

5.6 In binomial model, why the price of a derivative is irrelevant to the real world probability?

5.7 Options are redundant most of time, why trading options?

5.8 Volatility (standard deviation of the log price difference) and option price: There is a put maturing at 1 with strike price $K = 100$

(a) Market $\mathrm{BM}(S_0 = 100, u = 1.16, d = 0.87, p = 1/2, r = \ln(1.1))$.

(b) Market $\mathrm{BM}(S_0 = 100, u = 1.15, d = 0.85, p = 1/2, r = \ln(1.1))$.

Find the volatility and option price in each market, is there a simple monotonic relationship between volatility and put price?

5.9 In the market $\mathrm{BM}(S_0, u, d, p, r)$, there is a put maturing at 1 with strike price K, and $S_0u > K > S_0d$. Show that

(a) The price of the put P is increasing with u and decreasing with d.

(b) The price of the put P is increasing with $\sigma = \sqrt{\mathrm{var}(\ln(S_1/S_0))}$, the volatility of the stock price (standard deviation of the log price difference), if both $ud = c$ and p are constants. [Hint: $\frac{\mathrm{d}P}{\mathrm{d}\sigma} = \frac{\mathrm{d}P}{\mathrm{d}u}/\frac{\mathrm{d}\sigma}{\mathrm{d}u} > 0$ for $\frac{\mathrm{d}P}{\mathrm{d}u} = \frac{\partial P}{\partial u} + \frac{\partial P}{\partial d}\frac{\mathrm{d}d}{\mathrm{d}u} > 0$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}u} > 0$]

5.10 In the market $\mathrm{BM}(S_0, u, d, p, r)$, there is a call maturing at 1 with strike price K, and $S_0u > K > S_0d$. Show that

(a) The price of the call C is increasing with u and decreasing with d.

(b) The price of the call C is increasing with $\sigma = \sqrt{\mathrm{var}(\ln(S_1/S_0))}$, the volatility of the stock price (standard deviation of the log price difference), if both $ud = c$ and p are constants. [Hint: $\frac{\mathrm{d}C}{\mathrm{d}\sigma} = \frac{\mathrm{d}C}{\mathrm{d}u}/\frac{\mathrm{d}\sigma}{\mathrm{d}u} > 0$ for $\frac{\mathrm{d}C}{\mathrm{d}u} = \frac{\partial C}{\partial u} + \frac{\partial C}{\partial d}\frac{\mathrm{d}d}{\mathrm{d}u} > 0$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}u} > 0$]

5.11 In $\mathrm{BM}_3(S_0, u, d, p, r)$ market, let's define $q = \frac{e^r - d}{u - d}$ and

$$\begin{aligned}
\mathrm{Q}(HHH) &= q^3 \\
\mathrm{Q}(HHL) &= q^2(1-q) \\
\mathrm{Q}(HLH) &= aq^2(1-q) \\
\mathrm{Q}(HLL) &= q(1-q)(1-aq) \\
\mathrm{Q}(LHH) &= (2-a)q^2(1-q) \\
\mathrm{Q}(LHL) &= q(1-q)(1-(2-a)q) \\
\mathrm{Q}(LLH) &= q(1-q)^2 \\
\mathrm{Q}(LLL) &= (1-q)^3
\end{aligned}$$

If Q is a probability measure, show that $(\max(0, 2 - 1/q) < 1 < \min(1/q, 2))$

(a) $\max(0, 2 - 1/q) \leqslant a \leqslant \min(1/q, 2)$

(b) $\mathrm{Q}(w_{t+1} = H \mid S_t) = q$ for $t = 0, 1, 2$

§ 5.5 Appendix

some

5.5.1 Stochastic Processes

When t represents time, we can interpret $f(t,\omega_i)$ and $f(t,\omega_j)$ as two different trajectories that depend on different states of the world. For example, in Figure 5.3, $\omega_3 = LHL$ marks the path of down-up-down, the collection of paths $\{\omega_1, \omega_2, \cdots, \omega_8\}$ is the sample space Ω. Hence, if ω represents the underlying randomness, the function $f(t,\omega)$ can be called a *random function*. Another name for random functions is *stochastic processes*. With stochastic processes, t will represent time, and we often limit our attention to the set $t \geqslant 0$.

Note this fundamental point. Randomness of a stochastic process is in terms of the trajectory as a whole, rather than a particular value at a specific point in time. In other words, the random drawing is done from a collection of trajectories Ω. Choosing the state of the world, $\omega \in \Omega$, determines the complete trajectory.

The price of stock at time t will be denoted by $S(t)$.

$$S(t) : \Omega \to (0, \infty)$$

We shall write $S(t,\omega)$ to denote the price at time t if the market follows scenario $\omega \in \Omega$.

Remark: at time t, ω is not fully revealed, infinite path into the future.

A *predictable (previsible) process* is one which only depends on information available up to the current time, but not on any information in the future. Formally, $X_t \in \mathbb{I}_t$.

Remark: most textbook define the discrete previsible process by $X_t \in \mathbb{I}_{t-1}$, a predictable process is "known one step ahead in time". They use X_t in the place of my X_{t-1}.

In the $\mathrm{BM}_T(S_0, u, d, p, r)$ market, $\omega = w_1 w_2 \cdots w_T$, the stock price process S_t is predictable, $S_t(w_1 w_2 \cdots w_T)$ is resolved by the first t coin tosses $w_{1:t} = w_1 w_2 \cdots w_t$, $S_t(w_1 w_2 \cdots w_T)$ is not dependent on $w_{t+1:T} = w_{t+1} w_{t+2} \cdots w_T$. For example

$$S_2(LH) = S_2(LHH) = S_2(LHLLL)$$

Conversely, knowing the value of S_t at time t gives no hints of $w_{t+1:T}$ on the path of $w_{1:T}$.

5.5.2 Martingale

A martingale is the probabilistic extension of a flat line.

A martingale is a stochastic process for which, at a particular time, the expectation of the future value is equal to the present observed value even given knowledge of full past history.

If random process $Y_t \in \mathbb{I}_t$, and for all t and u with $t < u$ the following relation holds:

$$\mathrm{E}_t(Y_u) = \mathrm{E}(Y_u \mid \mathbb{I}_t) = Y_t$$

then we say that stochastic process $Y(t)$ is an $(\mathbb{I}_t)$-martingale.

It is important to note that the property of being a martingale involves both the filtration and the probability measure (with respect to which the expectations are taken). It is possible that Y could be a martingale with respect to one measure but not another one;

Remark: $\mathbb{I}_t$ information up to t, called filtration, The information generated by X_t on the interval $[0, t]$, $X_s \in \mathbb{I}_t = \mathbb{I}_{X_t}$ for any $s \leqslant t$.

Doob's Optional Stopping Theorem, which says that the expectation of a martingale is constant, even if we stop the martingale at a random time (so long as that random time does not look into the future).

In mathematical finance and economics, martingales are crucial for pricing models. For example, if we model a financial asset as a random process, we demand pricing rules (measures) under which the asset is a martingale. The martingality of an asset is equivalent to not being able to conduct arbitrage through trades in that asset.

5.5.3 Proof

Let $\mathsf{L}^p(\Omega, \mathcal{F}, \mathrm{P})$ be the collection of random variables such that $\mathrm{E}(|X|^p) < \infty$ for any random variable X and real number $p \geqslant 1$. Since payoffs must be finite, we limit our discussion of random variables with finite expectations. In binomial model, it is clear that $\mathsf{L}^1(\Omega, \mathcal{F}, \mathrm{P}) = \mathsf{L}^2(\Omega, \mathcal{F}, \mathrm{P})$, thus both are Hilbert spaces.

A: Free of Arbitrage

Theorem 5.2: $\mathrm{BM}(S_0, u, d, p, r)$ is free of arbitrage iff $d < e^r < u$.

Proof: free of arbitrage $\Longrightarrow$ (5.2). To show that absence of arbitrage implies (5.2), we assume that (5.2) does in fact not hold, and then we show that this implies an arbitrage opportunity.

If (5.2) fails, either $e^r \geqslant u$ or $e^r \leqslant d$. Let us thus assume that one of the inequalities in (5.2) does not hold, so that we have, say, the inequality $e^r \geqslant u > d$, we have

$$e^r - Z \gneqq 0$$

So it is always more profitable to invest in the bond than in the stock. An arbitrage strategy is now formed by the portfolio $\mathbf{h} = [-1; S_0]$, i.e. we sell the stock short and invest all the money in the bond. For this portfolio we obviously have $V_0 = -1 \cdot S_0 + S_0 \cdot 1 = 0$

$$V_1 = -1 \cdot S_0 Z + S_0 \cdot e^r = S_0\left(e^r - Z\right) \gneqq 0$$

the case $e^r \leqslant d < u$ is treated similarly.

What about "(5.2) $\Longrightarrow$ absence of arbitrage"?

Otherwise, we can find an arbitrage portfolio $[a; b]$

$$V_0 = aS_0 + b = 0$$

$$V_1 = aS_0 Z + be^r \gneq 0$$

which means $b = -aS_0$ and

$$V_1 = aS_0\,(Z - e^r) \gneq 0$$

Note that $a \neq 0$, otherwise $V_1 = 0$. Therefore, either $a < 0$ or $a > 0$. Assume now that $a > 0$, then

$$V_1 \gneq 0 \implies V_1 \geqslant 0 \implies Z \geqslant e^r \implies d \geqslant e^r$$

contradict with $d < e^r$. Similarly, $a < 0$ leads to contradiction. $\square$

Discuss: if condition (5.2) is true, define $q = \frac{e^r - d}{u - d}$, then $X_0 = e^r\,\mathrm{E}^Q(X)$ for B and S, it is a pricing function, and rule out arbitrage.

B: Pricing Formula

Theorem 5.7: $X_0 = e^r\,\mathrm{E}^Q(X)$, the pricing function takes a model specific form

free of arbitrage $\implies$ (5.2), define $q = \frac{e^r - d}{u - d}$ and $1 - q$ to be measure Q. By Eq (5.6), $X_0 = e^r\,\mathrm{E}^Q(X)$

$\impliedby$, for any $V_1 \gneq 0$, $\mathrm{E}^Q(V_1) > 0$, thus $V_0 = e^r\,\mathrm{E}^Q(V_1) > 0$, rule out arbitrage opportunity

C: No-arbitrage and Completeness

Proposition 5.12: $\mathrm{BM}_T(S_0, u, d, p, r)$ is free of arbitrage if and only if $d < e^r < u$.

$\implies$: each node is one-step tree, interpret q as probability, $V_t = e^{-r}\,\mathrm{E}_t^Q(V_{t+1}) \implies V_0 = e^{-rT}\,\mathrm{E}^Q(V_T)$, any $V_T \gneq 0$, $V_0 > 0$

$\impliedby$: otherwise $d < e^r < u$ not hold, find an arbitrage in one-step model, and convert to risk-free asset and hold to T.

Proposition 5.13: $\mathrm{BM}_T(S_0, u, d, p, r)$ is complete

Proof: one-step is complete, condition on $t = T-1, T-2, \cdots, 1, 0$, working backward on a non-recombining tree

D: Q World

If we define probability measure Q by

$$\mathrm{Q}(\,w_{t+1} = H \,|\, \mathbb{I}_t) = q \qquad t = 0, 1, \cdots, T-1$$

then

$$\mathrm{Q}(w_1 w_2 \cdots w_T) = q^n (1-q)^{T-n}$$

In the following proof, the notation $\mathrm{Q}(w_1 w_2 \cdots w_T)$ is somewhat misused. It is a shorthand, as an example, let $T = 3$, and $w_1 w_2 w_3 = HLH$, then $\mathrm{Q}(w_1 w_2 w_3)$ refers to $\mathrm{Q}(HLH)$ or $\mathrm{Q}(w_1 = H, w_2 = L, w_3 = H)$, and $\mathrm{Q}(w_2 \,|\, w_1)$ is $\mathrm{Q}(w_2 = L \,|\, w_1 = H)$.

Proof: by the Proposition 3.14, $\{w_t = H\} \perp \mathbb{I}_{t-1}$ and $\{w_t = L\} \perp \mathbb{I}_{t-1}$, thus

$$\mathrm{Q}(w_1 w_2 \cdots w_T) = \mathrm{Q}(w_1)\mathrm{Q}(\,w_2 \,|\, w_1) \cdots \mathrm{Q}(\,w_T \,|\, w_1 w_2 \cdots w_{T-1}) = \mathrm{Q}(w_1)\mathrm{Q}(w_2) \cdots \mathrm{Q}(w_T)$$

Bibliography

Capiński, Marek and Tomasz Zastawniak, 2003. *Mathematics for Finance: An Introduction to Financial Engineering*. Springer-Verlag, London, UK

Hull, John C., 2014. *Options, Futures and Other Derivatives*, 9/e. Prentice Hall, Boston