§ 7.1 General Properties of Option Price

sorry, in draft state

factors affecting the price of a stock option

7.1.1 Effect of Variables on Option Pricing

See Hull 10e, p232 Table 11.1

The following variables may affect the price of a stock option: Let t be the current time, and T be the expiration date.

1. The strike price, K
2. The time to expiration, $T-t$
3. The current stock price, S_t
4. The annualized expected return on stock in a short period, μ (percentage drift rate)
5. The volatility of the stock price per year, σ (percentage volatility)
6. The risk-free interest rate, r
7. The dividends that are expected to be paid.

We assume that μ and $\sigma > 0$ are constant, and $r \geqslant 0$ is constant and the same for all maturities.

Remark: more exactly, the expected return μ is a drift parameter, it is an *annualized log expected return*

$$\mu = \frac{\ln(\mathrm{E}(S_T/S_t))}{T-t}$$

Recall that if $\ln(S_T/S_t) \sim \mathrm{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$, then the annualized expected *log return* is

$$\frac{\mathrm{E}(\ln(S_T/S_t))}{T-t} = \mu - \frac{\sigma^2}{2}$$

However, we have $\mathrm{E}(S_T/S_t) = e^{\mu(T-t)}$.

Remark: Jensen's inequality shows that $\ln(\mathrm{E}(S_T/S_t)) > \mathrm{E}(\ln(S_T/S_t))$

In an infinitesimal time interval Δt

$$\mathrm{E}\left(\frac{\Delta S_t}{S_t}\right) = e^{\mu\Delta t} - 1 = \mu\Delta t$$

$$\mathrm{var}\left(\frac{\Delta S_t}{S_t}\right) = \left(e^{\sigma^2\Delta t} - 1\right) e^{2(\mu-\sigma^2/2)\Delta t + \sigma^2\Delta t} = \sigma^2\Delta t$$

in a very short period of time, the mean return is $\mu\Delta t$. Note that approximately $\frac{\Delta S_t}{S_t} \sim \mathrm{N}(\mu\Delta t, \sigma^2\Delta t)$ (Hull 10e, p320, Eq15.1) is not true, but $\frac{\Delta S_t}{S_t}$ is shifted lognormal!

$$\frac{\Delta S_t}{S_t} = \frac{S_{t+\Delta t}}{S_t} - 1 = \exp\left(\sigma\Delta W_t + \left(\mu - \frac{\sigma^2}{2}\right)\Delta t\right) - 1$$

$\frac{\Delta S_t}{S_t} > -1$, the stock is a limited liablity.

A: Strike Price

Let C_t be the price of a European call, show that C_t is a decreasing, convex function of K. And for $h > 0$, $C_t(S,T,K) - C_t(S,T,K+h) \leqslant e^{-r(T-t)}h$ (We do not assume the BS market)

Let $C_t(K) = C_t(S,T,K)$ then

$$C_T(K) = (S_T - K)^+$$

and $C_t = \wp_{t,T}(C_T)$, where the pricing function is a positive linear function.

Decreasing: $C_T(K)$ is a decreasing function of K ($a > b \implies a^+ \geqslant b^+$). If $K_1 > K_2$, then $C_T(K_1) \leqslant C_T(K_2)$ and

$$C_t(K_1) = \wp(C_T(K_1)) \leqslant \wp(C_T(K_2)) = C_t(K_2)$$

Convex: $P_T(K)$ is a convex function of K (Lemma 7.14). Let $0 \leqslant l \leqslant 1$, $K = lK_1 + (1-l)K_2$, then

$$lC_T(K_1) + (1-l)C_T(K_2) \geqslant C_T(K)$$

and

$$C_t(K) = \wp(C_T(K)) \leqslant \wp(lC_T(K_1) + (1-l)C_T(K_2)) = lC_t(K_1) + (1-l)C_t(K_2)$$

By Triangle Inequality ($a = S_T - (K+h)$, $b = h > 0$)

$$C_T(K) - C_T(K+h) \leqslant h^+ = h$$

or by $a > b \implies a^+ - b^+ \leqslant a - b$

$$C_T(K) - C_T(K+h) \leqslant (S_T - K) - (S_T - (K+h)) = h$$

thus

$$C_t(K) - C_t(K+h) \leqslant \wp_{t,T}(h) = e^{-r(T-t)}h$$

Let P_t be the price of a European put, show that P_t is an increasing, convex function of K. And for $h > 0$, $P_t(S,T,K+h) - P_t(S,T,K) \leqslant e^{-r(T-t)}h$ (We do not assume the BS market)

See Exercise 7.8.

B: Put-call Parity

When the prices of European put and call options have the same strike price and time to maturity, there is an important relationship

$$P_t + S_t = C_t + Ke^{-r(T-t)}$$

A European call is a portfolio of a corresponding European put and a forward

Only need linearity, The law of linear combination asserts that the pricing function is linear, do not need no-arbitrage assumption. The law of one price and law of linear combination result in $\wp(0) = 0$, which rules out immediate arbitrage opportunities.

Put-call parity holds only for European options. However, it is possible to derive some results for American option prices

7.1.2 Upper and Lower Bounds for Option Prices

Assume $0 \leqslant r < \infty$, then $0 < \wp(1) = e^{-r} \leqslant 1$

Upper and lower bounds for European call options

$$S_t > C_t \geqslant (S_t - Ke^{-r(T-t)})^+$$

for European put options

$$Ke^{-r(T-t)} > P_t \geqslant (Ke^{-r(T-t)} - S_t)^+$$

Proof. For any real number a, b, and c

$$\max(a, b) \geqslant a \quad \max(a, b) \geqslant b$$

and

$$c \geqslant a,\ c \geqslant 0 \iff c \geqslant a^+$$

and $a > 0, b > 0 \implies (a-b)^+ < a$

✓European call: For $S_T > 0$ and $K > 0$, $C_T = (S_T - K)^+ < S_T$, thus

$$C_t = \wp(C_T) < \wp(S_T) = S_t$$

$C_T = (S_T - K)^+$, then $C_T \geqslant 0$ and $C_T \geqslant S_T - K$, by the positivity of pricing function $C_t = \wp(C_T) \geqslant 0$ and $\wp(C_T) \geqslant \wp(S_T - K) = S_t - Ke^{-r(T-t)}$, thus $C_t \geqslant (S_t - Ke^{-r(T-t)})^+$.

✓European puts: For $S_T > 0$ and $K > 0$, $P_T = (K - S_T)^+ < K$, thus

$$P_t = \wp(P_T) < \wp(K) = Ke^{-r(T-t)}$$

$P_T = (K - S_T)^+$, then $P_T \geqslant 0$ and $P_T \geqslant K - S_T$, by the positivity of pricing function $P_t = \wp(P_T) \geqslant 0$ and $\wp(P_T) \geqslant \wp(K - S_T) = Ke^{-r(T-t)} - S_t$, thus $P_t \geqslant (Ke^{-r(T-t)} - S_t)^+$. □

Remark: By parity $P_t = C_t - (S_t - Ke^{-r(T-t)}) \geqslant 0$ and $C_t = P_t - (Ke^{-r(T-t)} - S_t) \geqslant 0$ means $C_t \geqslant S_t - Ke^{-r(T-t)}$ and $P_t \geqslant Ke^{-r(T-t)} - S_t$.

Since $C_T = (S_T - K)^+ \gneq 0$, $C_t = \wp(C_T) > 0$. Similarly, $P_t > 0$.

$$0 < C_t < S_t \qquad 0 < P_t < K$$

When $r > 0$, since $C_t \geqslant S_t - Ke^{-r(T-t)}$, there is $C_t > S_t - K$.

7.1.3 American Options

No matter what happens, an American call option can never be worth more than the stock. Otherwise, an arbitrageur could easily make a riskless profit by buying the stock and selling the call option. An American put option gives the holder the right to sell one share of a stock for K. Similarly, no matter how low the stock price becomes, the option can never be worth more than K. We have the upper bounds (the value of a right to obtain an asset can not exceed the asset itself)

$$C_A(t) < S_t \qquad P_A(t) < K$$

Proof. $C_A(t) < S_t$: At time t, hold portfolio $X_t = S_t - C_A(t)$. If C_A is exercised at time $u < T$, $X_u = S_u - (S_u - K) = K$, we hold Ke^{-ru} unit bonds to time T (self-financing), then $X_T = Ke^{r(T-u)} > 0$. At time T, whether it is exercised or not, $X_T = S_T - (S_T - K)^+ = \min(K, S_T) > 0$. Thus $S_t - C_A(t) = X_t = \wp(X_T) > 0$, that is, $C_A(t) < S_t$. One can argue by arbitrage too.

$P_A(t) < K$: At time t, hold portfolio $X_t = K - P_A(t)$. If P_A is exercised at time $u < T$, $X_u = Ke^{r(u-t)} - (K - S_u) = S_u + K(e^{r(u-t)} - 1)$, we hold $(S_u + K(e^{r(u-t)} - 1))e^{-ru}$ unit bonds to time T (self-financing), then $X_T = (S_u + K(e^{r(u-t)} - 1))e^{r(T-u)} > 0$. At time T, whether it is exercised or not, $X_T = K - (K - S_T)^+ = \min(K, S_T) > 0$. Thus $K - P_A(t) = X_t = \wp(X_T) > 0$, that is, $P_A(t) < K$ □

An American option gives at least the same rights as the corresponding European option. Although may not be wise, exercising and holding to maturity are the two easy actions among all the trading strategies at time t:

- $C_A(t) \geqslant \max(S_t - K, C_E(t))$, thus $C_A(t) \geqslant C_E(t)$ and $C_A(t) \geqslant S_t - K$
- $P_A(t) \geqslant \max(K - S_t, P_E(t))$, thus $P_A(t) \geqslant P_E(t)$ and $P_A(t) \geqslant K - S_t$

Remark: The price of an American option must be not less than the values of actions can be taken now. Otherwise, that we long the option and take the action properly will result in an arbitrage opportunity. For example, if $C_A(t) < S_t - K$, we long the call and exercise it immediately, the immediate profit will be $S_t - K - C_A(t) > 0$ without future obligation. If $P_A(t) < P_E(t)$, long P_A and short P_E and hold P_A to maturity, $X_t = P_A(t) - P_E(t) < 0$, and $X_T = P_A(T) - P_E(T) = 0$, there is an arbitrage opportunity.

The price of an American option equals the highest value of actions can be taken now. One of the partition of action sets is to exercise or not, thus $C_A(t) = \max(S_t - K, C_A(t + 0))$ (the second item $C_A(t + 0)$ means that it is not exercised at time t).

A: Early Exercise

It shows that it is never optimal to exercise an American call option on a non-dividend-paying stock prior to the option's expiration (if $r > 0$), but that under some circumstances the early exercise of an American put option on such a stock is optimal. When there are dividends, it can be optimal to exercise either calls or puts early.

Since $r > 0$

$$C_A(t) \geqslant C_E(t) \geqslant (S_t - Ke^{-r(T-t)})^+ \geqslant S_t - Ke^{-r(T-t)} > S_t - K$$

the investor is better off selling the option than exercising it. Thus

$$C_A(t) = C_E(t)$$

for an American call option on a non-dividend-paying stock.

There are two reasons an American call on a non-dividend-paying stock should not be exercised early: One relates to the insurance that it provides against the stock price falling below the strike price (pay less). The other reason concerns the time value of money. From the perspective of the option holder, the later the strike price is paid out the better (pay later).

If $r = 0$, there is $P_A(t) = P_E(t)$.

Since $P_A(T) = (K - S_T)^+ \gtrsim K - S_T$. If early exercise at time $u < T$, the payoff is

$$K - S_u = \wp(K - S_T) < \wp(P_A(T))$$

Which means that it is better not to exercise early. Thus $P_A(t) = P_E(t)$ if $r = 0$.

Remark: When the interest rates are not deterministic, even if $R_{t,T} = 0$, we do not have $R_{u,T} = 0$, then $K - S_t = \wp_{t,T}(K - S_T)$, but $K - S_u \neq \wp_{u,T}(K - S_T) = e^{-(T-u)R_{u,T}}K - S_u$.

B: Put-call Inequality

The prices of American put and call options with the same strike price X and expiry time T on a stock that pays no dividends satisfy

$$S(t) - Ke^{-r(T-t)} \geqslant C_A(t) - P_A(t) \geqslant S(t) - K$$

Proof. The left inequality: by $C_A = C_E$, and $P_A \geqslant P_E$

$$\begin{aligned} P_A(t) + S(t) &\geqslant P_E(t) + S(t) && P_A \geqslant P_E \\ &= C_E(t) + Ke^{-r(T-t)} && \text{parity} \\ &= C_A(t) + Ke^{-r(T-t)} && C_A = C_E \end{aligned}$$

The right inequality: At time t, long $C_A + Ke^{-rt} \cdot e^{rt}$ (one American call and Ke^{-rt} unit bonds), short $P_A + S$.

- If the put option is exercised early at time $t < u < T$

$$\begin{aligned} X_u &= C_A(u) + Ke^{r(u-t)} - (K - S_u + S_u) \\ &= C_A(u) + Ke^{r(u-t)} - K > 0 \qquad C_A(u) = C_E(u) > 0 \end{aligned}$$

  We sell C_A and convert all proceeds into bonds (self-financing), then $X_T = X_u e^{r(T-u)} > 0$.
- If the put is not early exercised (the put option is exercised at time T, or not exercised at all), we find that at time T

$$\begin{aligned} X_T &= C_A(T) + Ke^{r(T-t)} - [P_A(T) + S(T)] \\ &= (S_T - K)^+ + Ke^{r(T-t)} - \left[(K - S_T)^+ + S_T\right] \\ &= Ke^{r(T-t)} - K \geqslant 0 \end{aligned}$$

Thus, in all circumstances, there always be $X(t) = \wp(X_T) \geqslant 0$, that is $C_A(t) - P_A(t) \geqslant S(t) - K$. □

Remark: possibility of equality

- If the put option is exercised at time T, or not exercised at all, there is $C_A(T) = C_E(T)$ and $P_A(T) = P_E(T)$, we have $C_A(T) - P_A(T) = S(T) - K$.

- When r is very small $C_A(t) - P_A(t) \approx S(t) - K$, it is approximately equal.

7.1.4 Effect of Dividends

to do: effect of dividends

in a deterministic world, then uncertainty is introduced

7.2.1 Deterministic Model

Suppose percentage change of stock price is proportional to small time span

$$\frac{\Delta S_t}{S_t} = \frac{S_{t+\Delta t} - S_t}{S_t} = \mu \Delta t \implies \frac{\Delta S_t}{\Delta t} = \mu S_t$$

in the limit as $\Delta t \to 0$

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = \mu S_t$$

so that

$$S_t = S_0 e^{\mu t}$$

where S_0 and S_t are the stock price at time zero and time t. The stock price grows at a continuously compounded rate of μ. This is like the risk-free bond B_t!

Remark

- S_t is function of t, note that we also denote it by $S(t)$
- taking logarithm

  $$\ln(S_t) = \ln(S_0) + \mu t$$

  the log price grows linearly

7.2.2 Geometric Brownian Motion

If $Z \sim \mathrm{N}(0,1)$, we define

$$S_1 = S_0 e^{\mu} e^{cZ} \implies \ln(S_1) = \ln(S_0) + \mu + cZ$$

where c is a constant to be determined. This model is more realistic, since it contains random factors. However, the term e^{cZ} produces the other source of drift. We wish to keep all the drift in the e^{μ} factor. Note that

$$\mathrm{E}\left(e^{cZ}\right) = e^{c^2/2} \implies \mathrm{E}(e^{cZ - c^2/2}) = \mathrm{E}\left(e^{cZ}\right) e^{-c^2/2} = 1$$

we can normalize our model to correct for this unwanted drift

$$\ln(S_1) = \ln(S_0) + \mu + cZ - c^2/2$$

Suppose each period has length $\Delta t = h$, let us fix a time $t = Nh$ for some integer N, denote

$$S_n = S(nh) \qquad n = 0, 1, 2, \cdots, N$$

Let $\{Z_n\}$ denote a sequence of i.i.d standard normal random variables with mean 0 and variance 1. We set

$$\ln(S_n) = \ln(S_{n-1}) + \mu h + cZ_n - c^2/2 \tag{7.1}$$

where c is a constant and $c^2/2$ is to correct the drift. Then we have

$$\ln(S_n) = \ln(S_0) + \mu n h + cR_n - nc^2/2$$

where

$$R_n = Z_1 + Z_2 + \cdots + Z_n \sim \mathrm{N}\left(0, n\right)$$

The stock price

$$S_n = S_0 e^{n\mu h} e^{cR_n} e^{-nc^2/2}$$

Remark

- S_0 is simply the initial price of the stock ($t = 0$)
- $e^{n\mu h}$ is the drift or deterministic component
- e^{cR_n} is random factor, $R_n \sim \mathrm{N}\left(0, n\right)$
- $e^{-nc^2/2}$ is correction factor

Remind: If $\{X_n\}$ is a sequence of iid $\mathrm{N}\left(\mu, \sigma^2\right)$, then $X_1 + X_2 + \cdots + X_n \sim \mathrm{N}\left(n\mu, n\sigma^2\right)$

We choose a relationship between c and N so that

$$\mathrm{var}\left(cR_N\right) = \sigma^2 t$$

thus

$$\sigma^2 t = \mathrm{var}\left(cR_N\right) = c^2 \mathrm{var}\left(R_N\right) = c^2 N \implies c = \sigma\sqrt{h}$$

Because

$$S_N = S\left(Nh\right) = S\left(t\right) = S_t = S_0 e^{\mu N h} e^{cR_N} e^{-Nc^2/2}$$

let's denote

$$W_t \equiv \frac{1}{\sigma} cR_N = \sqrt{h} R_N = \sqrt{\Delta t} R_N \sim \mathrm{N}\left(0, t\right)$$

we have

$$S_t = e^{\mu t} S_0 e^{\sigma W_t} e^{-\sigma^2 t/2} = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

which is a lognormal model, call geometric Brownian motion (GBM).

Remark: Let

$$W_n = \frac{c}{\sigma} R_n = \sqrt{\Delta t} R_n$$

we have (forward difference)

$$\Delta W_n = W_{n+1} - W_n = Z_{n+1}\sqrt{\Delta t}$$

symbolically in stochastic calculus

$$\mathrm{d}W = \epsilon\sqrt{\mathrm{d}t} \qquad \epsilon \sim \mathrm{N}(0, 1)$$

and

$$\left(\mathrm{d}W\right)^2 = \epsilon^2 \mathrm{d}t \to \mathrm{d}t$$

by $\mathrm{E}\left(\epsilon^2\right) = 1$, $\mathrm{E}\left(\epsilon^4\right) = 3$

$$\mathrm{E}((\mathrm{d}W)^2) = \mathrm{d}t \qquad \mathrm{var}((\mathrm{d}W)^2) = \mathrm{E}((\mathrm{d}W)^4) - [\mathrm{E}((\mathrm{d}W)^2)]^2 = 2 \cdot (\mathrm{d}t)^2 \to 0$$

The variance of $(\Delta W)^2 = \epsilon^2 \Delta t$ is therefore too small for it to have a stochastic component. As a result, $\left(\mathrm{d}W\right)^2 = \mathrm{d}t$, we can treat $\epsilon^2 \Delta t$ as nonstochastic and equal to its expected value, Δt, as Δt tends to zero. Note again that all the reasoning above has been purely motivational.

Try ($o(\Delta t)$ means $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}=0$)

$$\begin{aligned}\frac{\Delta S_t}{S_t} &= \frac{S_{t+\Delta t}-S_t}{S_t} = \frac{S_{t+\Delta t}}{S_t}-1 = \frac{S_0\exp\left(\sigma W_{t+\Delta t}+\left(\mu-\frac{\sigma^2}{2}\right)(t+\Delta t)\right)}{S_0\exp\left(\sigma W_t+\left(\mu-\frac{\sigma^2}{2}\right)t\right)}-1\\
&= \exp\left(\sigma\Delta W_t+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right)-1\\
&= 1+\sigma\Delta W_t+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\frac{1}{2}\left(\sigma\Delta W_t+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t\right)^2+o(\Delta t)-1\\
&= \sigma\Delta W_t+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\frac{1}{2}\sigma^2(\Delta W_t)^2+\left(\mu-\frac{\sigma^2}{2}\right)\sigma\Delta W_t\Delta t+o(\Delta t)\\
&= \sigma\Delta W_t+\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\frac{1}{2}\sigma^2\Delta t+\left(\mu-\frac{\sigma^2}{2}\right)\sigma\epsilon\cdot(\Delta t)^{3/2}+o(\Delta t)\\
&= \mu\Delta t+\sigma\Delta W_t+o(\Delta t)\end{aligned}$$

or

$$\frac{\mathrm{d}S}{S}=\mu\mathrm{d}t+\sigma\mathrm{d}W$$

rules: like $\mathrm{i}^2=-1$, we write $(\mathrm{d}W)^2=\mathrm{d}t$

When randomness is introduced, the rules are changing, the rules working well in deterministic world fail in a random world. A bike need a rack to stand up, however, when riding, we do not need the rack.

Remark: the traditional (deterministic) limit can not be use, for $\epsilon_t\sim\mathrm{N}(0,1)$ is random

$$\lim_{\Delta t\to 0}\frac{\Delta W}{\Delta t}=\lim_{\Delta t\to 0}\frac{\epsilon_t\sqrt{\Delta t}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\epsilon_t}{\sqrt{\Delta t}}=\infty \quad a.s.$$

To realize the above argument on a computer take e.g. $\Delta t=10^{-20}$. Then $\Delta W(t)/\Delta t=\epsilon_t/\sqrt{\Delta t}=10^{10}\epsilon_t$, which is very large in absolute value with overwhelming probability.

Question: Why forward difference?

math consideration, to be independent of past

economic meaning, future innovation (information)

If using backward difference, $\Delta W_t=W_t-W_{t-\Delta t}$, at time t, ΔW_t is known, there is nothing random.

Equation (7.1) gives

$$\ln(S_{n+1})=\ln(S_n)+\mu h+cZ_{n+1}-c^2/2=\ln(S_n)+\mu\Delta t+\sigma\Delta W_n-\sigma^2\Delta t/2$$

or

$$\mathrm{d}\ln(S)=(\mu-\sigma^2/2)\mathrm{d}t+\sigma\mathrm{d}W$$

Note that $\mathrm{d}\ln(S)\neq\frac{\mathrm{d}S}{S}$

7.2.3 Binomial Model to Geometric Brownian Motion

We divide a time period from 0 to t into N subintervals, each has length $h = t/N$. Let $S_N = S_0 Z_1 Z_2 \cdots Z_N$ with (We are interest in Q world for derivative pricing)

$$\mathrm{Q}(Z_n = u) = q \qquad \mathrm{Q}(Z_n = d) = 1 - q$$

Note that u, d and q will depend on N (or sub-period length h).

Remark: $Z_1, Z_2, \cdots, Z_N$ are i.i.d, and we can work in P world similarly.

A: Log Price

To match mean and variance

$$\ln(Z_n) = \ln(S_n/S_{n-1}) \longleftrightarrow \mathrm{N}^Q(\gamma h, \sigma^2 h)$$

with $\gamma = r - \frac{\sigma^2}{2}$ (where r is annual rate, not period rate), let

$$q\ln(u) + (1-q)\ln(d) = \gamma h$$
$$q(1-q)(\ln(u) - \ln(d))^2 = \sigma^2 h$$

For simplicity (symmetry), assume that $ud = 1$, then

$$q = \frac{\gamma h - \ln(d)}{\ln(u) - \ln(d)}$$

and

$$\ln(u) = \sqrt{\gamma^2 h^2 + \sigma^2 h} \qquad \ln(d) = -\ln(u)$$

We see that

$$q = \frac{\gamma h - \ln(d)}{\ln(u) - \ln(d)} = \frac{\gamma h + \ln(u)}{2\ln(u)} = \frac{\gamma}{2}\sqrt{\frac{h}{\gamma^2 h + \sigma^2}} + \frac{1}{2}$$

Define

$$Y_n = \mathit{1}_{Z_n = u} - \mathit{1}_{Z_n = d} = \begin{cases} 1 & q \\ -1 & 1-q \end{cases}$$

If U is the number of times the stock goes up by time t and D is the number of times the stock price goes down by time t then obviously, we must have

$$U - D = \sum_{n=1}^{N} Y_n$$

then

$$S_t = S_N = S_0 u^U d^D = S_0 u^{U-D} = S_0 \exp\left(\sqrt{\gamma^2 h + \sigma^2} \cdot \sqrt{h} \sum_{n=1}^{N} Y_n\right)$$

It can be shown that (see the appendix)

$$\lim_{h \to 0} \sqrt{\gamma^2 h + \sigma^2} \cdot \sqrt{h} \sum_{n=1}^{N} Y_n \sim \mathrm{N}^Q(\gamma t, \sigma^2 t) \tag{7.2}$$

thus

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \qquad W_t \sim \mathrm{N}^Q(0, t)$$

B: Discussion

safe to skip this subsection

An alternative way to find the q, u and d is by matching

$$Z_n = S_n/S_{n-1} \sim \mathrm{LN}^Q((r-\sigma^2/2)h, \sigma^2 h)$$

with

$$qu + (1-q)d = e^{rh}$$
$$q(1-q)(u-d)^2 = (e^{\sigma^2 h} - 1)e^{2rh}$$

and again assume that $ud = 1$, then [find q from (1), substitute in (2) and simplify]

$$u + \frac{1}{u} = e^{(r+\sigma^2)h} + e^{-rh} \equiv c$$

which gives

$$u = \frac{c + \sqrt{c^2-4}}{2} \qquad d = \frac{1}{u} = \frac{c - \sqrt{c^2-4}}{2}$$

and

$$q = \frac{e^{rh}-d}{u-d} = \frac{e^{rh}-c/2}{\sqrt{c^2-4}} + \frac{1}{2} = \frac{2e^{rh} - (e^{(r+\sigma^2)h} + e^{-rh})}{2\sqrt{(e^{(r+\sigma^2)h} + e^{-rh})^2 - 4}} + \frac{1}{2} \tag{7.3}$$

It can be shown that

$$\lim_{N\to\infty} q = \frac{1}{2}$$

and (Exercise)

$$\lim_{N\to\infty} N^{1/2}(2q-1) = t^{1/2}\frac{\gamma}{\sigma}$$

However, the limit distribution of

$$S_t = S_0 u^{U-D} = S_0 \left(\frac{c+\sqrt{c^2-4}}{2}\right)^{\sum_{n=1}^N Y_n}$$

is hard to work out.

How about

$$\ln(S_t/S_0) = \ln\left(\frac{c+\sqrt{c^2-4}}{2}\right)\sum_{n=1}^N Y_n$$

In most applications, we set

$$u = e^{\sigma\sqrt{h}} \qquad d = e^{-\sigma\sqrt{h}} \tag{7.4}$$

and $q = \frac{e^{rh}-d}{u-d}$, then

$$S_t = S_0 u^{U-D} = S_0 e^{\sigma\sqrt{h}\sum_{n=1}^N Y_n}$$

It can be shown that

$$\lim_{h\to 0} \sigma\sqrt{h}\sum_{n=1}^N Y_n \sim \mathrm{N}^Q(\gamma t, \sigma^2 t) \tag{7.5}$$

thus $S_t/S_0 \sim \mathrm{LN}^Q(\gamma t, \sigma^2 t)$.

Remark: In simple method, we approximate $u = \frac{c+\sqrt{c^2-4}}{2}$ by $u = e^{\sigma\sqrt{h}}$, the variance are not exact match in one period (but exact in the limit)

$$q(1-q)(u-d)^2 = (e^{\sigma^2 h} - 1)e^{2rh} + o(h^2)$$

for (by small o)

$$\begin{aligned}
&q(1-q)(u-d)^2 - (e^{\sigma^2 h}-1)e^{2rh} \\
&= \frac{e^{rh}-d}{u-d}\left(1-\frac{e^{rh}-d}{u-d}\right)(u-d)^2 - (e^{\sigma^2 h}-1)e^{2rh} \\
&= (e^{rh}-d)(u-e^{rh}) - e^{\sigma^2 h+2rh} + e^{2rh} \\
&= e^{hr}(e^{\sigma\sqrt{h}} + e^{-\sigma\sqrt{h}}) - 1 - e^{\sigma^2 h+2rh} \\
&= (1+hr+o(h^2))\cdot(2+\sigma^2 h+o(h^2)) - 1 - (1+\sigma^2 h+2rh+o(h^2)) \\
&= o(h^2)
\end{aligned}$$

When h is small, $\sqrt{h}$ much larger than h

Remark: fixed $q = 1/2$, by

$$\gamma h = \frac{1}{2}(\ln(u)+\ln(d)) \qquad \sigma^2 h = \frac{1}{4}(\ln(u)-\ln(d))^2$$

then $\ln(u) = h\gamma + \sqrt{h}\sigma$, $\ln(d) = h\gamma - \sqrt{h}\sigma$, the problem is

$$\mathrm{E}^Q(Y_n) = 2q-1 = 0$$

not able to generate the drift term

§ 7.3 Black-Scholes Model

The Black-Scholes model has two assets, a risk-free bond with price process B_t and a (risky) stock with price process S_t defined as

$$B_t = e^{rt}$$
$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

Where $0 \leqslant t \leqslant T$, W_t is a Wiener process. The (percentage) drift μ, the (percentage) volatility $\sigma > 0$, the continuously compounded risk-free rate $r > 0$, and the initial stock price $S_0 > 0$ are constants. The prices process of stock S_t is called a geometric Brownian motion. Modelling by stochastic calculus, the price dynamics are

$$B_t = B_0 + \int_0^t rB_u \, \mathrm{d}u \qquad B_0 = 1$$
$$S_t = S_0 + \int_0^t \mu S_u \, \mathrm{d}u + \int_0^t \sigma S_u \, \mathrm{d}W_u$$

or symbolically, in differential form

$$\frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t \qquad \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

The market is driven by a Wiener process, we define the sample space naturally as

$$\Omega = \{w(t) : w \in \mathsf{C}[0, T]\}$$

Where $\mathsf{C}[0, T]$ is the collection of continuous functions define on interval $[0, T]$, thus each continuous function is an elementary event. Let $\mathbb{I}_t$ be the information available up to time t (the information generated by W_s on the interval $[0, t]$, $\mathbb{I}(W_t) = \mathbb{I}(S_t)$ and $\mathbb{I}_0$ contains only sets of measure one and sets of measure zero), then the event space is $\mathcal{F} = \mathbb{I}_T$. For brevity, the market setting of Black-Scholes model is denoted by $\mathrm{BS}_T(r, \mu, \sigma)$.

7.3.1 Pricing Function: Q World

Let $\lambda = \frac{\mu - r}{\sigma}$ and

$$G = \exp\left(-\int_0^T \lambda(u) \, \mathrm{d}W(u) - \frac{1}{2}\int_0^T \lambda^2(u) \, \mathrm{d}u\right)$$

then by Girsanov's Theorem, probability measure Q given by $\mathrm{d}Q = G\mathrm{d}P$ is equivalent to P, and

$$Z(t) = W(t) + \int_0^t \lambda(u) \, \mathrm{d}u$$

is a Wiener process under Q.

If we normalized the asset prices by the price of risk-free bond B_t, we define

$$S_{:t} = S_t / B_t \qquad B_{:t} = B_t / B_t = 1$$

We say that B_t is the *numéraire*, the asset in which values of other assets are measured. For convenience, this price system is called B *price system*. By Itô formula, there is

$$\mathrm{d}S_{:t} = \sigma S_{:t} \mathrm{d}Z_t$$

Which means that $S_{:t}$ is a Q martingale. In more details, for any t and $u \geqslant t$

$$\mathrm{E}_t^Q(S_{:u}) = S_{:t} \qquad 0 \leqslant t \leqslant u \leqslant T$$

When $0 \leqslant t \leqslant u \leqslant T$, we have

$$S_t = B_t \, \mathrm{E}_t^Q(S_u/B_u)$$
$$B_t = B_t \, \mathrm{E}_t^Q(B_u/B_u)$$

We see that $V_t = B_t \, \mathrm{E}_t^Q(V_u/B_u)$ works for all primary assets, thus

$$V_t = \wp_{t,u}(V_u) \equiv B_t \, \mathrm{E}_t^Q(V_u/B_u) = e^{-r(u-t)} \, \mathrm{E}_t^Q(V_u) \tag{7.6}$$

is a pricing function in the $\mathrm{BS}_T(r, \mu, \sigma)$ market.

7.3.2 No-arbitrage and Completeness

Let us first specify our vocabulary in the $\mathrm{BS}_T(r, \mu, \sigma)$ market. Let

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

be a partition of $[0, T]$. Think of $t_0, t_1, \cdots, t_{N-1}$ as the trading dates, and think of

$$[a_{t_i}; b_{t_i}] \in \mathbb{I}_{t_i} \qquad i = 0, 1, 2, \cdots, N-1$$

as the position (number of shares) taken in stock and bond at each trading date and held to the next trading date. If each rebalancing is self-financing, then

$$a_{t_{i-1}} S_{t_i} + b_{t_{i-1}} B_{t_i} = a_{t_i} S_{t_i} + b_{t_i} B_{t_i}$$

and

$$\begin{aligned} \Delta V_{t_i} = V_{t_{i+1}} - V_{t_i} &= a_{t_{i+1}} S_{t_{i+1}} + b_{t_{i+1}} B_{t_{i+1}} - [a_{t_i} S_{t_i} + b_{t_i} B_{t_i}] \\ &= a_{t_i} S_{t_{i+1}} + b_{t_i} B_{t_{i+1}} - [a_{t_i} S_{t_i} + b_{t_i} B_{t_i}] = a_{t_i} \Delta S_{t_i} + b_{t_i} \Delta B_{t_i} \end{aligned}$$

for $i = 1, 2, \cdots, N-1$. If the mesh of partition go to zero ($\max(t_{i+1} - t_i) \to 0$), in the Itô sense, we have the *self-financing condition*

$$\mathrm{d}V_t = a_t \mathrm{d}S_t + b_t \mathrm{d}B_t \tag{7.7}$$

where

$$\int_0^T \mathrm{E}(a^2(u) + b^2(u)) \, \mathrm{d}u < \infty \tag{7.8}$$

Condition (7.8) is a technical requirement, which makes sure that the corresponding Itô integral is a martingale, and rules out possible doubling strategies for a sure win.

Definition 7.1: In the $\mathrm{BS}_T(r, \mu, \sigma)$ market, a *self-financing trading strategy* (*portfolio strategy*) is a stochastic process

$$\mathbf{h}_t = [a_t; b_t] \in \mathbb{I}_t \qquad t \in [0, T]$$

satisfy condition (7.7) and (7.8). The value process of portfolio $\mathbf{h}_t$ is

$$V_t = V(\mathbf{h}_t) = a_t S_t + b_t B_t$$

The portfolio we buy at time t is allowed to depend on all information up to time t, by observing the evolution of the stock price. We are, however, not allowed to look into the future. Such trading strategy is said to be *adapted* or *predictable*.

Proposition 7.2: A stochastic process $\{\mathbf{h}_t = [a_t; b_t]\}$ is a self-financing portfolio strategy if and only if $\mathrm{d}V_{:t} = a_t \mathrm{d}S_{:t}$.

Proof. For $V_t = V(\mathbf{h}_t) = a_t S_t + b_t B_t$ and

$$\mathrm{d}S_{:t} = \mathrm{d}\left(\frac{S_t}{B_t}\right) = \frac{1}{B_t}(\mathrm{d}S_t - S_t r \mathrm{d}t)$$

$$\mathrm{d}V_{:t} = \mathrm{d}\left(\frac{V_t}{B_t}\right) = \frac{1}{B_t}(\mathrm{d}V_t - V_t r \mathrm{d}t)$$

We have

$$\begin{aligned}(\mathrm{d}V_{:t} - a_t \mathrm{d}S_{:t})B_t &= (\mathrm{d}V_t - V_t r\mathrm{d}t) - a_t(\mathrm{d}S_t - S_t r\mathrm{d}t) \\ &= \mathrm{d}V_t - a_t\mathrm{d}S_t - (V_t - a_t S_t) r\mathrm{d}t \\ &= \mathrm{d}V_t - (a_t \mathrm{d}S_t + b_t \mathrm{d}B_t)\end{aligned}$$

Thus $\mathrm{d}V_{:t} = a_t \mathrm{d}S_{:t} \iff \mathrm{d}V_t = a_t\mathrm{d}S_t + b_t\mathrm{d}B_t$. □

An arbitrage portfolio is defined by condition (1.13). With respect to $\mathrm{BS}_T(r, \mu, \sigma)$ market, an arbitrage possibility is a self-financing process $V_t = V(\mathbf{h}_t)$ with the following properties

$$V_0 = 0, \ V_T \gneq 0$$

In $\mathrm{BS}_T(r, \mu, \sigma)$ market, given r, the parameters μ and σ have three cases: (1) $\sigma > 0$, (2) $\sigma = 0$ and $\mu = r$, and (3) $\sigma = 0$ and $\mu \neq r$. The following Proposition answers why we assume $\sigma > 0$ in the $\mathrm{BS}_T(r, \mu, \sigma)$ market.

Proposition 7.3: In $\mathrm{BS}_T(r, \mu, \sigma)$ market, the market is free of arbitrage if and only if the following condition holds: either $\sigma > 0$, or $\sigma = 0$ with $\mu = r$.

Proof. $\implies$: If $\sigma = 0$ with $\mu = r$, then $S_t = B_t$, there is only one risk-free asset, $V_t = \wp_{t,u}(V_u) = e^{-r(u-t)}V_u$; If $\sigma > 0$, the pricing formula is (7.6), $V_t = \wp_{t,u}(V_u) = e^{-r(u-t)}\,\mathrm{E}_t^Q(V_u)$. Thus, there always exits a positive linear pricing function $V_t = \wp_{t,u}(V_u)$. For any self-financing process V_t with $V_T \gneq 0$, $V_0 = \wp_{0,T}(V_T) > 0$, the market is free of arbitrage.

$\impliedby$: otherwise, the market is free of arbitrage and $\sigma = 0$ with $\mu \neq r$, then the stock is riskless, $S_t = S_0 e^{\mu t}$, we have

$$S_0 = \wp(S_T) = \wp(S_0 e^{\mu T}) = S_0 e^{\mu T} \wp(1) = S_0 e^{(\mu - r)T}$$

which requires $\mu = r$, contradiction with $\mu \neq r$. □

Since the primary assets B_t and S_t are in L^2, the payoff space of self-financing portfolio strategies must be in $\mathsf{L}^2_T(\Omega, \mathcal{F}, \mathrm{P})$. If all T-claims ($X \in \mathbb{I}_T$ with finite variance, European style with payoff X at time T) can be replicated by a self-financing portfolio strategy we say that the market is complete.

Proposition 7.4: The $\mathrm{BS}_T(r, \mu, \sigma)$ market is complete, i.e. every T-claim can be replicated by a self-financing portfolio strategy.

Proof. Given T-claim $X \in \mathbb{I}_T$, let $X_t = B_t \, \mathrm{E}_t^Q(X/B_T)$, then $X_T = X$ and $X_{:t} = X_t/B_t$ is a Q martingale. By Martingale Representation Theorem, there exists a unique predictable process h_t such that

$$\mathrm{d}X_{:t} = h_t \mathrm{d}Z_t = a_t \mathrm{d}S_{:t}$$

where $a_t = \frac{h_t B_t}{\sigma S_t}$. Let $b_t = \frac{X_t - a_t S_t}{B_t}$, following from Proposition 7.2, $X_t = a_t S_t + b_t B_t$ is a self-financing portfolio. Thus, every T-claim can be replicated by a self-financing portfolio strategy. □

Risk Neutral Valuation: The $\mathrm{BS}_T(r, \mu, \sigma)$ market is complete, every T-claim X can be replicated by a self-financing portfolio strategy $\{[a_t; b_t]\}$ such that $X_t = a_t S_t + b_t B_t$ and

$$\frac{X_t}{B_t} = \mathrm{E}_t^Q\left(\frac{X}{B_T}\right)$$

7.3.3 Black-Scholes Equation

Knowing that S_t is a Markov process in P world and in Q world, given a simple T-claim $X = f(S_T)$

$$X_t = B_t \, \mathrm{E}^Q\left(\frac{X}{B_T} \middle| \mathbb{I}_t\right) = B_t \, \mathrm{E}^Q\left(\frac{X}{B_T} \middle| S_t\right) = e^{-r(T-t)} \, \mathrm{E}^Q\left(f(S_T) \,|\, S_t\right)$$

Which shows that the price of any simple T-claim $X = f(S_T)$ is a function of t and S_t, which can be denoted by $f(t, S_t)$. By Feynman-Kac formula

$$f(t, S_t) = e^{-r(T-t)} \, \mathrm{E}^Q\left(f(S_T) \,|\, S_t\right)$$

is the solution to the following PDE (partial differential equation)

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{7.9}$$

the celebrated Black-Scholes equation.

By the completeness of $\mathrm{BS}_T(r, \mu, \sigma)$ market, any simple T-claim $X = f(S_T) = f(T, S_T)$ can be replicated by a self-financing portfolio $\{[a_t; b_t]\}$, and (drop the subscript for clarity)

$$\mathrm{d}f = a\mathrm{d}S + b\mathrm{d}B$$

or

$$\left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t + \sigma S\frac{\partial f}{\partial S}\mathrm{d}W = a\mu S\mathrm{d}t + a\sigma S\mathrm{d}W + bBr\mathrm{d}t$$

thus

$$a = \frac{\partial f}{\partial S}$$

$$b = \frac{1}{Br}\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)$$

Since $aS + bB = f$, there is

$$\frac{\partial f}{\partial S} \cdot S + \frac{1}{Br}\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) \cdot B = f$$

Rearranging, we arrive at the Black-Scholes equation (7.9).

Remark: For a European call, the key financial insight behind the PDE is that one can perfectly hedge the option by buying and selling the underlying asset in just the right way and consequently "eliminate risk": The equation can be rewritten in the form

$$f - S\frac{\partial f}{\partial S} = \frac{1}{r}\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)$$

the left hand side equals $f - aS$, the right hand side equals bB, which is riskless. Thus, the Black-Scholes equation gives the details of the replicating portfolio.

Proposition 7.5: For any simple T-claim $f(T, S_T)$, the stochastic process $f(t, S_t)$ is a price process if and only if $f(t, S_t)$ satisfies the Black-Scholes equation (7.9).

Proof. Since $\mathrm{d}W = \mathrm{d}Z - \frac{\mu - r}{\sigma}\mathrm{d}t$

$$\begin{aligned}
\mathrm{d}f - rf\mathrm{d}t &= \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial S}(\mu S\mathrm{d}t + \sigma S\mathrm{d}W) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\mathrm{d}t - rf\mathrm{d}t \\
&= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} - rf\right)\mathrm{d}t + \mu S\frac{\partial f}{\partial S}\mathrm{d}t + \sigma S\frac{\partial f}{\partial S}\mathrm{d}W \\
&= \left(\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} - rf\right)\mathrm{d}t + \sigma S\frac{\partial f}{\partial S}\mathrm{d}Z
\end{aligned}$$

Thus, $f(t, S_t)$ is a price process $\iff \frac{f(t,S_t)}{B_t} = \mathrm{E}_t^Q\left(\frac{f(T,S_T)}{B_T}\right) \iff \mathrm{d}(f/B)$ is a Q martingale $\iff$ $\mathrm{d}(f/B) = (\mathrm{d}f - rf\mathrm{d}t)/B$ has no drift terms in Q world $\iff$ $f(t, S_t)$ satisfies the Black-Scholes equation. □

Proposition 7.5 validates the form $f(t, S_t)$ as the price process of European call assumed luckily in Black and Scholes (1973). Note that for non-simple T-claims, the price process may not take a form of $f(t, S_t)$. For example, the Asian option takes the form $f(t, S_t, A_t)$ where $A_t = \int_0^t S(u)\,\mathrm{d}u$.

§ 7.4 Black-Scholes Formula

Consider a European call option having strike price K and expiration time T. That is, the option allows one to purchase a single unit of an underlying security S at time T for the price K.

Call $C(S, T, K)$

Put $P(S, T, K)$

Suppose that the nominal interest rate is constant r, compounded continuously, such that the risk-free bond

$$B_t = e^{rt} \implies \frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t$$

and also that the price of the security follows a geometric Brownian motion (GBM) with drift parameter μ and volatility parameter σ:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \implies \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

where W_t is the standard Brownian motion in continuous time, $W_t \sim \mathrm{N}(0, t)$. Assuming that S_t follows GBM means that

- S_T has the lognormal distribution conditional on S_t
$$(S_T \mid S_t) \sim \mathrm{LN}\left(\ln(S_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$
equivalently $(\ln(S_T) \mid S_t) \sim \mathrm{N}\left(\ln(S_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$
- In Q world (risk neutral world), $\mathrm{d}S_{:t} = \sigma S_{:t}\mathrm{d}Z_t \implies \frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\mathrm{d}Z_t$ where Z_t is the standard Brownian motion in Q world. Hence, stock grows at risk free rate
$$(S_T \mid S_t) \sim \mathrm{LN}^Q\left(\ln(S_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right)$$
- S_t is a Markov process in P world and in Q world, which means $(S_T \mid \mathbb{I}_t) \sim (S_T \mid S_t)$
$$\mathrm{E}_t^Q(f(S_T)) = \mathrm{E}^Q(f(S_T) \mid \mathbb{I}_t) = \mathrm{E}^Q(f(S_T) \mid S_t)$$
for contract function $f(\cdot)$.

Remark: in Q world, $S_T \sim S_t e^{Z_{T-t}}$ with $Z_{T-t} \sim \mathrm{N}^Q((r - \sigma^2/2)(T-t), \sigma^2(T-t))$.

7.4.1 Risk Neutral Valuation

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by Eq (7.6): For any payoff X at time T, its value at time t is

$$\frac{X_t}{B_t} = \mathrm{E}_t^Q\left(\frac{X}{B_T}\right)$$

When we change the (annualized log) expected return of the underlying asset to be the risk-free interest rate, r (i.e., assume $\mu_t = r_t$)

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t \implies \frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\mathrm{d}Z_t$$

We effectively change the real world probability, the P world, into risk-neutral world, the Q world.

For the stock

$$\frac{S_t}{B_t} = \mathrm{E}_t^Q\left(\frac{S_T}{B_T}\right)$$

In the Black-Scholes market, r is a constant, $e^{-rt}S_t = \mathrm{E}^Q(e^{-rT}S_T \,|\, \mathbb{I}_t) = \mathrm{E}^Q(e^{-rT}S_T \,|\, S_t)$.

Theorem 7.6 (Risk Neutral Valuation): *In Black-Scholes market, the arbitrage free price of the simple T-claim $X = f(S_T)$ is given by*

$$X_t = e^{-r(T-t)}\,\mathrm{E}^Q\left(f(S_T)\,|\,S_t\right)$$

Binary options[1] are options with discontinuous payoffs. A simple example of a binary option is a cash-or-nothing call. A *standard cash-or-nothing call* pays off nothing if the stock price ends up below the strike price K at time T, and pays one dollar if it ends up above the strike price.

In the risk neutral world, the probability (conditional on current stock price S_t) that the stock price ends up above the strike price is (following from Eq 7.14)

$$\mathrm{Q}(S_T > K\,|\,S_t) = N(d_2)$$

with

$$d_2 = \frac{\ln(S_t/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

thus, the price of this cash-or-nothing call is

$$C_c(t) = e^{-r(T-t)}\,\mathrm{E}^Q\left(\mathit{1}_{S_T \geqslant K}\,|\,S_t\right) = e^{-r(T-t)}\mathrm{Q}\left(S_T \geqslant K\,|\,S_t\right) = e^{-r(T-t)}N(d_2)$$

Remark: it can be verify that $C_c(T) = \mathit{1}_{S_T \geqslant K}$ as $t \to T$ in $C_c(t)$

An *asset-or-nothing call* pays off nothing if the underlying stock price ends up below the strike price and pays an amount equal to the stock price itself if it ends up above the strike price.

For the asset-or-nothing call, the expectation of the payoff at T is, following from Eq (7.15)

$$\mathrm{E}^Q\left(S_T \cdot \mathit{1}_{S_T \geqslant K}\,|\,S_t\right) = S_t e^{r(T-t)}N(d_1)$$

with

$$d_1 = \frac{\ln\left(S_t/K\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Now we find the price of the asset-or-nothing call

$$C_a(t) = e^{-r(T-t)}\,\mathrm{E}^Q\left(S_T \cdot \mathit{1}_{S_T \geqslant K}\,|\,S_t\right) = S_t N(d_1)$$

7.4.2 European Call and Put Options

We are now ready to present the most celebrated formula in the theory of finance, the Black-Scholes option pricing formula

$$C_t = S_t N(d_1) - e^{-r(T-t)}KN(d_2) \tag{7.10}$$

[1] In contrast to ordinary financial options that typically have a continuous spectrum of payoff. The payoff of a binary option can take only two possible outcomes, either some fixed monetary amount (or a precise predefined quantity or units of some asset) or nothing at all. They are also called all-or-nothing options, digital options (more common in forex/interest rate markets), and fixed return options (FROs) (on the American Stock Exchange).

where

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

For the payoff of a European call at time T is

$$(S_T - K)^+ = (S_T - K) \cdot 1_{S_T \geqslant K} = S_T 1_{S_T \geqslant K} - K 1_{S_T \geqslant K}$$

The call option is decomposed into two options: short K standard cash-or-nothing and long an asset-or-nothing call. Finally, the price of the call is

$$C_t = C_a(t) - K \cdot C_c(t)$$

Remark: time is measured in year

The

$$C_t = C(t, S_t, T, K, \sigma, r)$$

is a function of six variables.

It follows that the price of the option depends on the underlying Brownian motion only through its volatility parameter σ and not its drift parameter μ.

Given r constant, option price is unchanged if the security's price over time is assumed to follow a geometric Brownian motion with a fixed volatility σ but with a drift that varies over time.

It follows from the put-call parity that the price of a European put option with initial price S_t, strike price K, and exercise time T, denoted by P_t, is given by

$$P_t = C_t + Ke^{-r(T-t)} - S_t = e^{-r(T-t)}KN(-d_2) - S_t N(-d_1) \tag{7.11}$$

Remark: Black-Scholes option pricing formula (7.10) can be solved by Black-Scholes equation (7.9) with boundary condition $f(T, S_T) = (S_T - K)^+$. The Feynman-Kac formula shows that these two methods lead to the same formula.

A: Properties of the Black-Scholes Formulas

What happens when some of the parameters take extreme values?

$$C_T = \lim_{t \to T} C_t = (S_T - K)^+$$

is the payoff at time T

When the stock price, S_t, becomes very large, a call option is almost certain to be exercised. It then becomes very similar to a forward contract with delivery price K.

$$S_t \to +\infty \implies C_t \to S_t - e^{-r(T-t)}K$$

$$P_t \to 0$$

this is consistent with the Black-Scholes Formulas, For

$$S_t \to \infty \implies d_1, d_2 \to +\infty \implies N(d_1), N(d_2) \to 1$$

What should the cost of a European call option become as the volatility becomes smaller and smaller? As volatility approaches zero, the stock is virtually riskless, its price will grow at risk-free rate r: At time

T and the payoff from a call option is

$$(S_t e^{r(T-t)} - K)^+$$

thus the value of the call today is the discounted value

$$(S_t - Ke^{-r(T-t)})^+$$

this is consistent with equation (7.10):

- If $S_t > Ke^{-r(T-t)}$, $C_t \to S_t - e^{-r(T-t)}K$. For

$$\sigma \to 0 \implies d_1, d_2 \to +\infty \implies N(d_1), N(d_2) \to 1$$

- If $S_t = Ke^{-r(T-t)}$, $C_t = 0$. For

$$\sigma \to 0 \implies d_1, d_2 \to 0 \implies N(d_1), N(d_2) \to 1/2$$

- If $S_t < Ke^{-r(T-t)}$, $C_t = 0$. For

$$\sigma \to 0 \implies d_1, d_2 \to -\infty \implies N(d_1), N(d_2) \to 0$$

B: At the Money

The price of an at-the-money call option on an asset is proportional to the asset price: By $K = S_t$

$$a_1 = \frac{(r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad a_2 = a_1 - \sigma\sqrt{T-t} = \frac{(r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

then

$$C_t = S_t N(a_1) - e^{-r(T-t)} K N(a_2) = C_t(1, T, 1) S_t \propto S_t$$

for

$$C_t(1, T, 1) \equiv N(a_1) - e^{-r(T-t)} N(a_2)$$

is a constant given the market setting of $\mathrm{BS}_T(r, \mu, \sigma)$, it does not depend on S_t.

Similarly, the price of an at-the-money put option on an asset is proportional to the asset price.

$$P_t = e^{-r(T-t)} K N(-a_2) - S_t N(-a_1) = [e^{-r(T-t)} N(-a_2) - N(-a_1)] S_t \propto S_t$$

7.4.3 Exotic Options

Application of BS formula

A: Chooser Option

The chooser option is an exotic option that gives the holder the right to choose, at some future date u, $0 < u < T$, between a European call and put written on the same underlying asset

1. Show that the value of the chooser option at time $t = u$ is

$$V_u = C(S_u, T, K) + (Ke^{-r(T-u)} - S_u)^+$$

2. Compute the price of chooser option at time $t = 0$

At $t = u$, let $C_u = C(S_u, T, K)$ and $P_u = P(S_u, T, K)$, the holder of the chooser option will pick the one with a higher value, thus

$$\begin{aligned} V_u &= \max(C_u, P_u) \\ &= \max(C_u, C_u + Ke^{-r(T-u)} - S_u) \\ &= C_u + (Ke^{-r(T-u)} - S_u)^+ \\ &= C(S_u, T, K) + P(S_u, u, Ke^{-r(T-u)}) \end{aligned}$$

the chooser option is a portfolio of a call option expiring at T with strike price K, and a put expiring at u with strike price $Ke^{-r(T-u)}$. Thus the price at $t = 0$ is

$$V_0 = \wp(V_u) = C\left(S_0, T, K\right) + P\left(S_0, u, Ke^{-r(T-u)}\right)$$

using BS formula to write out the detail.

B: Pay-later Option

A Pay-later option, also known as a contingent premium option, is a standard European option except that the buyer pays an amount A only at maturity of the option and if the option is in the money. The amount A is chosen so that the value of the option at time zero is zero. Find out the amount A in a pay-later call option.

This option is equivalent to a portfolio consisting of one standard European call option with strike K and maturity T, and short A digital call options (cash or nothing) with maturity T. The payoff at T is

$$X = \max\left(S_T - K, 0\right) - A \cdot 1\left(S_T \geqslant K\right)$$

At time 0

$$\begin{aligned} 0 = \wp(X) &= \wp(C(S_T, T, K)) - A\wp(C_c(T)) \\ &= C(S_0, T, K) - A \cdot C_c(0) \\ &= S_0 N\left(d_1\right) - e^{-rT} K N\left(d_2\right) - Ae^{-rT} N\left(d_2\right) \end{aligned}$$

thus

$$A = S_0 e^{rT} \frac{N\left(d_1\right)}{N\left(d_2\right)} - K$$

C: Ratchet Option

A two-leg ratchet call option can be described as follows. Two times u and T are fixed, $0 < u < T$. At time zero an initial strike price K is set. At time u the strike is reset to S_u. At the maturity time T the holder receives the payoff

$$(S_T - S_u)^+ + (S_u - K)^+$$

Compute the price of this option at $t < u$.

For

$$V_T = (S_T - S_u)^+ + (S_u - K)^+ = C(S_T, T, S_u) + C(S_u, u, K)$$

when $u \leqslant t \leqslant T$, $C(S_u, u, K) = (S_u - K)^+$ is a known number

$$\begin{aligned} V_t &= \wp(V_T) = \wp(C(S_T, T, S_u)) + \wp(C(S_u, u, K)) \\ &= C(S_t, T, S_u) + C(S_u, u, K)\wp(1) \\ &= C(S_t, T, S_u) + e^{-r(T-t)}C(S_u, u, K) \end{aligned}$$

note that at $t = u$ (at the money, call price proportional to stock price, $C(S_u, T, S_u) = C_u(1, T, 1)S_u$, and $C_u(1, T, 1)$ is a constant for fixed u and T)

$$V_u = C_u(1, T, 1)S_u + e^{-r(T-u)}C(S_u, u, K)$$

The value at $t < u$ is

$$\begin{aligned} V_t &= \wp(V_u) = \wp(C_u(1, T, 1)S_u) + \wp(e^{-r(T-u)}C(S_u, u, K)) \\ &= C_u(1, T, 1)\wp(S_u) + e^{-r(T-u)}\wp(C(S_u, u, K)) \\ &= C_u(1, T, 1)S_t + e^{-r(T-u)}C(S_t, u, K) \end{aligned}$$

§ 7.5 Exercise

7.1 Show that (Triangle Inequality)

$$(a+b)^+ \leqslant a^+ + b^+$$

and

$$\max(a+c, b+d) \leqslant \max(a,b) + \max(c,d)$$

7.2 Let $X \sim \mathrm{LN}(\mu, \sigma^2)$, and $a > 0$, compute

(a) $\mathrm{E}(\mathit{1}_{X \geqslant a})$, $\mathrm{E}(X \cdot \mathit{1}_{X \geqslant a})$ and $\mathrm{E}(X^2 \cdot \mathit{1}_{X \geqslant a})$.

(b) $\mathrm{E}(\mathit{1}_{X \leqslant a})$, $\mathrm{E}(X \cdot \mathit{1}_{X \leqslant a})$ and $\mathrm{E}(X^2 \cdot \mathit{1}_{X \leqslant a})$

7.3 Let $X \sim \mathrm{LN}(\mu, \sigma^2)$, compute $\mathrm{E}(X)$ and $\mathrm{var}(X)$.

7.4 Let $Y = \ln(X) \sim \mathrm{N}(\mu, \sigma^2)$, $\frac{\mathrm{d}Q}{\mathrm{d}P} = X/\mathrm{E}(X) = e^{Y-\mu-\sigma^2/2}$. Show that

(a) $Y \sim \mathrm{N}^Q(\mu+\sigma^2, \sigma^2)$

(b) For any $a > 0$, $\mathrm{Q}(X \geqslant a) = N\left(\frac{\mu - \ln(a)}{\sigma} + \sigma\right)$

(c) $\mathrm{E}(X \cdot \mathit{1}_{X \geqslant a}) = e^{\mu+\sigma^2/2}\mathrm{Q}(X \geqslant a)$

7.5 Given Eq (7.1), show that

$$S_n = S_0 e^{\mu n h} e^{c R_n} e^{-nc^2/2}$$

7.6 Given Eq (7.3), show that

$$\lim_{N \to \infty} q = \frac{1}{2}$$

and

$$\lim_{N \to \infty} N^{1/2}(2q-1) = t^{1/2}\frac{\gamma}{\sigma}$$

7.7 Show that $C_A(t) \geqslant C_E(t)$ and $P_A(t) \geqslant P_E(t)$.

7.8 Let P_t be the price of a European put, show that P_t is an increasing, convex function of K. And for $h > 0$, $P_t(S,T,K+h) - P_t(S,T,K) \leqslant e^{-r(T-t)}h$ (We do not assume the BS market)

7.9 A forward contract delivered at T, is made at time t on the claim X. Let interest rate r be constant and $t < U < T$

(a) Find the value for this contract at time U

(b) Find the value for this contract at time U if $X = S_T$

7.10 Let the intrinsic value of a call at time t be

$$V_t = V(S_t) = (S_t - K)^+$$

where $0 \leqslant t \leqslant T$ and $K > 0$.

(a) Show that when $x \geqslant 0$ and $0 \leqslant l \leqslant 1$, $V(lx) \leqslant lV(x)$

(b) Show that $e^{-rt}V_t$ is a submartingale ($\mathrm{E}_t(X_u) \geqslant X_t, t < u$) under Q in the Black-Scholes market

(c) Let C_t be the price of European call, show that $C_t \geqslant V_t$ in the Black-Scholes market

7.11 In the Black-Scholes market, a cash-or-nothing call pays off nothing if the stock price ends up below the strike price K at time T, and pays a fixed amount, one dollar, if it ends up above the strike price. Find the price of the call.

7.12 In the Black-Scholes market, find the price of asset-or-nothing call, which pays off nothing if the underlying stock price ends up below the strike price and pays an amount equal to the stock price itself if it ends up above the strike price.

7.13 Golden Logarithm: A contingent claim of the form $X = \ln(S_T)$. Note that if $S_T < 1$, this means that the holder has to pay a positive amount. Please find the price for this derivative in the Black-Scholes market.

7.14 Ratio derivative: Two times $0 < u < T$ are fixed. The derivative matures at time T with payoff S_T/S_u. Find the value of the derivative at times $t < u$ in the Black-Scholes market.

7.15 Let S_t be price of a stock, and an option with payoff at T is $S_T^{1/3}$. Find the price of the option at time t in the Black-Scholes market.

7.16 Let S_t be price of a stock, and an option with payoff at T is S_T^2. Find the price of the option at time t in the Black-Scholes market.