CHAPTER 4

Interest Rates

Interest is the amount of money paid by a borrower to the owner as a form of compensation for the use of *principal* (the amount of money borrowed). Interest is generally calculated as a percentage of the principal sum per year, which percentage is known as an *interest rate*.

In the setting of simple interest, interest must be insulated from the principal, there is no interest on interest. However, rather than paying it out or keeping apart, interest can be reinvested to accumulate further interest. Compound interest is standard in finance and economics, interest in the next period is then earned on the principal sum plus previously accumulated interest.

The future spot rate is stochastic in nature. The relationship between the forward rate and future spot rate is investigate first in a time-varying but deterministic setting, it is straightforward that the future spot rate must be the forward rate. However, in a stochastic setting where the market is free of arbitrage, the expectation of future spot rate is not even equal to the forward rate.

§ 4.1 Compounding

Compounding is the process of generating earnings on an asset's reinvested earnings. Albert Einstein called compound interest "the greatest mathematical discovery of all time". In a deterministic world, we explain and compare various compounding methods, and relate interests to rate of returns.

4.1.1 Simple Interest

To begin with, we shall consider the case when the amount of interest is proportional to the length of time. Assume that the interest rate r is constant, for an initial deposit as the principal P, after one year the interest earned will be rP, and $2rP$ for a two year deposit.

Rule of Simple Interest

The value of the investment at time t, denoted by $V(t)$, is given by

$$V(t) = P + t \cdot rP = (1 + rt)P \qquad t \geqslant 0 \tag{4.1}$$

where time t is expressed in years.

Example 4.1.1: Consider a deposit of \$250 held for 30 days at a simple interest rate of 8%. This gives $P = 250$, $t = 30/365$ and $r = 0.08$. After 30 days the deposit will grow to

$$P + t \cdot rP = \left(1 + \frac{30}{365} \cdot 0.08\right) \cdot 250 = 251.64$$

Remark: a year will have 365 days if the particular year number is not given.

In practice simple interest is used only for short-term investments and for certain types of loans and deposits. Because interest is not added to the principal, it is not a realistic description of the value of money in the longer term. In the majority of cases the interest already earned can be reinvested to attract even more interest.

4.1.2 Periodic Compounding

Suppose that an amount P is invested for one year at an interest rate of r per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$(1 + r)P$$

When the interest rate is compounded twice per annum, the semiannual compounding means that $\frac{r}{2}P$ is earned for the first 6 months. At the middle of the year, the interest is then added on to principal, $P + \frac{r}{2}P = P \cdot \left(1 + \frac{r}{2}\right)$ is regarded as the new principal for another six-month at rate $r/2$. In this case, with the interest being reinvested, the initial amount P grows to

$$P \cdot \left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) = \left(1 + \frac{r}{2}\right)^2 P$$

Given a natural number m, if the rate is compounded m times per annum, it is equivalent to paying interest rate at $\frac{r}{m}$ for each $1/m$ years and subsequently compounded (per $1/m$ years, compounded every

$1/m$ years). The first interest being due at time $1/m$ is $\frac{r}{m}P$, and the principal increases to $(1 + r/m)P$. At time $2/m$, the principal updates to $(1 + r/m)^2 P$. With the interest being reinvested again and again until year-end, the terminal value of the investment is

$$\left(1 + \frac{r}{m}\right)^m P$$

In these circumstances, the interest earned will now be added to the principal periodically, the interest rate is measured by *discrete* or *periodic compounding*.

Rule of Compound Interest

At an interest rate of r per annum, if the rate is compounded m times per annum, the future value at time t of an initial principal P will become

$$V(t) = \left(1 + \frac{r}{m}\right)^{mt} P \tag{4.2}$$

The interest earned by an investment according to simple interest is the same in one given time period as in any other time period. In contrast, compound interest is earned not only on principal (as in the case of simple interest) but also on all interest that has already been earned.

Example 4.1.2: The effect of increasing the compounding frequency on the value of $\$100$ at the end of one year when the interest rate is 10% per annum

Compounding Frequency	m	Value at End of One Year
Annually	$m = 1$	110.00
Semiannually	$m = 2$	110.25
Quarterly	$m = 4$	110.38
Monthly	$m = 12$	110.47
Weekly	$m = 52$	110.51
Daily	$m = 365$	110.52

We see that 10.47% with annual compounding is equivalent to 10% with monthly compounding.

Analogy: For the four components in formula (4.2), the principal P is your body, the interest rate r is your growing potential, compounding frequency m is your devotion to life, exercise, study and work, and time t is your health. To maximize your future value, you should first keep your body safe, develop your potential and live a longer time. Starting early is helpful to reach the same goal. What is most important is your devotion, more devotions also improve your P, r and t.

Remark: Formula (4.2) works when future time t is a multiple of $1/m$ years. In fact, it is accepted and applied to any time $t \geqslant 0$, for it is theoretically consistent in perfect market.

- Most interest rates are quoted on an annual basis. Thus, if the time interval is not specified, the default interval is one year.
- When interest rates are quoted per period, it will be simply compounded (once per period) if the compounding frequency is omitted. For example, given an interest rate at 10 basis points per day

without giving the compounding frequency, this periodic rate is assumed to be daily compounded.

Example 4.1.3: *The Doubling Rule*: If you put funds into an account that pays interest at rate r compounded annually, how many years does it take for your funds to double?

We need to find the value of t such that $V_t = 2P$ with $m = 1$, say

$$(1 + r)^t = 2$$

thus

$$t = \frac{\ln(2)}{\ln(1 + r)} \approx \frac{0.7}{r}$$

For instance, if the interest rate is 1% ($r = 0.01$) then it will take approximately 70 years for your funds to double; If $r = 0.05$, it will take about 14 years; If $r = 0.07$, it will take about 10 years; and if $r = 0.10$, it will take about 7 years.

As a check on the preceding approximations, note that (to four decimal place):

$$1.01^{70} = 2.0068 \qquad 1.05^{14} = 1.9799$$
$$1.07^{10} = 1.9672 \qquad 1.10^{7} = 1.9487$$

4.1.3 Continuous Compounding

The limit as the compounding frequency, m, tends to infinity is known as *continuous compounding*. With continuous compounding, it can be shown that an amount P invested for t years at rate r grows to

$$V(t) = e^{rt} P \tag{4.3}$$

where e is the base of the natural logarithm. The fact that the rate is continuously compounded implies that the calculation of terminal value will be made using the equation (4.3).

Example 4.1.4: Growing with continuous compounding at an interest rate of 10%, the initial value of \$100 at the end of one year is $e^{0.1} \cdot 100 = 110.52$. Which is (to two decimal places) the same as the value with daily compounding.

In the case of continuous compounding

$$\frac{\mathrm{d}V}{\mathrm{d}t} = rV(t) \tag{4.4}$$

the rate of change is proportional to the current wealth. The percentage changes in value, $\frac{\mathrm{d}V}{V} = r\mathrm{d}t$, is approximately equal to interest rate in a very short period of time. Because the continuous compounding is clean and more convenient in calculus, to keep the mathematics simple, interest rates in this book will be measured with continuous compounding except where stated otherwise.

By Equation (4.3), given present value V_t at time t, the future value at time T is

$$V_T = V_t e^{r(T-t)} \tag{4.5}$$

and $\wp_{t,T}(1) = e^{-r(T-t)}$. Thus given future value V_T, the present value at time t is

$$V_t = \wp_{t,T}(V_T) = e^{-r(T-t)} V_T \tag{4.6}$$

4.1.4 Compare Compounding Methods

A statement by a bank that the interest rate on one-year deposits is 6% sounds straightforward and unambiguous. In fact, its precise meaning depends on the way the interest rate is measured. In Example 4.1.2, we notice that more frequent compounding will produce a higher future value than less frequent compounding if the interest rates and the initial principal are the same. Thus, to accurately define the amount to be paid under a legal contract with interest, the frequency of compounding (yearly, semiannually, quarterly, monthly, weekly, daily, etc.) and the interest rate must be specified.

A: Effective Rate

Let us define the *effective interest rate* $r_{:}$ by

$$1 + r_{:} = V_1/V_0 = V_1/P$$

We see that the effective interest rate $r_{:}$ equals the amount earned from an investment of \$1 for one year. It is an equivalent annual interest rate, also called *annual equivalent rate* (AER), or shortly *effective rate*. For a given compounding method with an interest rate of r per annum

1. When simple interest is used, the effective rate is $r_{:s} = r$.
2. When compounding with frequency m, the effective rate is
$$r_{:m} = \left(1 + \frac{r}{m}\right)^m - 1$$
3. With continuous compounding, the effective rate is
$$r_{:c} = e^r - 1$$

We can think of the difference between one compounding frequency and another to be analogous to the difference between kilometers and miles. They are just two different units of measurement.

We say that two compounding methods are *equivalent* if the corresponding effective interest rates are the same. If one of the effective interest rate exceeds the other, then the corresponding compounding method is said to be *preferable*.

Example 4.1.5: Daily compounding at 15%, monthly compounding at 15.1%, and semi-annual compounding at 15.5%, which one is preferable?

The effective rate for daily compounding at 15% is

$$r_{:365} = \left(1 + \frac{0.15}{365}\right)^{365} - 1 = 16.18\%$$

similarly, $r_{:12} = 16.19\%$ and $r_{:2} = 16.10\%$. Thus, daily compounding at 15% is preferable to semi-annual compounding at 15.5%, and monthly compounding at 15.1% is most preferable.

Suppose that r_c is a rate of interest with continuous compounding and r_m is the equivalent rate with compounding m times per annum. By the definition of annual equivalent rate, we have

$$e^{r_c} = \left(1 + \frac{r_m}{m}\right)^m$$

This means that

$$r_c = m \ln\left(1 + \frac{r_m}{m}\right) \tag{4.7}$$

or

$$r_m = (e^{r_c/m} - 1)m \tag{4.8}$$

These formulas can be used to convert a rate continuously compounded to a rate with a compounding frequency of m times per annum, and vice versa.

Example 4.1.6: Suppose that a lender quotes the interest rate on loans as 8% with continuous compounding, and that interest is actually paid quarterly. Find out the quarterly interest payments on a $1,000 loan.

The equivalent rate with quarterly compounding is

$$r_4 = (e^{0.08/4} - 1) \cdot 4 = 8.08\%$$

thus interest payments of $\$1000 \cdot r_4/4 = \$20.20$ would be required each quarter.

Suppose that the interest rate r is constant, compounded m times per annum, then equation (4.2) is correct for any time $t \geqslant 0$, not just for whole multiples of $1/m$. Due to the bridge of continuous compounding, $r_c = m\ln\left(1 + \frac{r}{m}\right)$, thus

$$V(t) = e^{r_c t}P = e^{tm\ln\left(1+\frac{r}{m}\right)}P = \left(1 + \frac{r}{m}\right)^{mt} P$$

Remark: For fixed deposit, if it is cashed out early, the bank will apply the interest rate of the current account, not of the fixed account.

Example 4.1.7: Suppose that a CD promising to pay $120 after one year is priced at $100. If an investor decided to sell such a CD half a year before maturity, what is the fair price?

It depends on the interest rate at that time. We assume the interest rate r is a constant, given $V_1 = 120$ and $P = 100$

$$r = \ln(V_1/P) = \ln(120/100) = 18.23\%$$

and at $t = 0.5$

$$V_t = e^{-r(1-t)}V_1 = (e^r)^{-(1-t)}V_1 = 1.2^{-0.5} \cdot 120 = 109.54$$

A frequent first guess is $110, based on halving the annual profit of $20. It is possible that the fair price is 110, if $V_t = 110$, by Eq (4.5)

$$r = \frac{\ln(V_1/V_t)}{1-t} = \frac{\ln(120/110)}{1-0.5} = 17.40\%$$

which means that the interest rate jumps down to 17.40%.

B: Periodic Rate

The *periodic rate* is the amount of interest that is charged for each period divided by the amount of the principal. The *nominal annual rate* or *nominal interest rate* is commonly defined as the periodic rate multiplied by the number of periods per year. For example, a monthly rate of 1% is equivalent to an annual nominal interest rate of 12%, irrelevant of the compounding method.

The precise meaning of an interest rate r requires two ingredients: time unit and compounding period:

1. Time unit: one day, one month, usually one year. The interest rate r is quoted for this particular time span. Let the time unit be h years, then one year contains $1/h$ periods.
2. Compounding period: if $m_p \geqslant 1$ is the number of compounding per period (time unit), h/m_p is the compounding period. For periodic rate, if the compounding period is not explicitly stated, *simple compounding* (once per period, $m_p = 1$) is assumed.

For an interest rate of r_p *per period* (periodic rate) compounded m_p times per period, the future value Eq (4.2) takes the following form

$$V_p(n) = \left(1 + \frac{r_p}{m_p}\right)^{m_p n} P \tag{4.9}$$

where n is the number of periods. Following the usual market practice

- The nominal annual rate: $r = r_p \cdot (1/h) = r_p/h$
- The compounding frequency per annum: $m = m_p \cdot (1/h) = m_p/h$
- The effective rate is

$$r_{:p} = \left(1 + \frac{r_p}{m_p}\right)^{m_p/h} - 1$$

Please note that, no matter which compounding method is used, the periodic rate r_p equals nominal annual rate r times length of period h

$$r_p = r \cdot h$$

Since $mh = m_p$, let $t = nh$, given nominal annual rate r, we see that the future value

$$V(t) = V(nh) = V_p(n) \tag{4.10}$$

for any compounding method (Exercise 4.6). Whatever it is compounded m_p times per period ($V_p(n)$ by Eq 4.9), continuously compounded ($V_p(n) = e^{r_p n} P$), or never compounded (simple interest rate, $V_p(n) = (1 + r_p n)P$). Thus, an interest rate of r per annum is equivalent to an interest rate of $r_p = rh$ per period under the same compounding method.

Remark: There is a reinvestment assumption. However, most of time, the interest rate for a certain period is not able to roll forward.

Example 4.1.8: Given a quarterly rate of 3% with monthly compounded, what is

(a) The nominal annual rate with annual compounding?

(b) Effective rate?

(c) Equivalent rate per annum with continuous compounding?

Given $r_p = 3\%$, $h = 1/4$ and $m_p = 3$. The nominal annual rate is

$$r = r_p \cdot (1/h) = 12\%$$

irrelevant of the compounding frequency. The effective rate is

$$r_{:p} = \left(1 + \frac{3\%}{3}\right)^{3/(1/4)} - 1 = 12.68\%$$

For the equivalent rate per annum with continuous compounding, by Eq (4.7)

$$r_c = m \ln\left(1 + \frac{r}{m}\right) = 3/(1/4) \cdot \ln\left(1 + \frac{12\%}{3/(1/4)}\right) = 11.94\%$$

or by the same effective interest rates $r_{:p} = r_{:c} = e^{r_c} - 1$, there is $r_c = \ln(r_{:p} + 1) = 11.94\%$.

4.1.5 Rate of Return

Given a value process $V(t)$, if $V(s) > 0$, the simple rate of return (or simply *return*) over a period of time from s to t is defined by

$$Y(s,t) = \frac{V(t) - V(s)}{V(s)} = \frac{V_t}{V_s} - 1 \tag{4.11}$$

The return is realized at time t. If $V(s) = 0$, it is a zero investment, return is not defined. If $V(s) < 0$, the return computed by Eq (4.11) gives an opposite feeling. In a short sale[1], for example, assume $V_0 = -7$ and $V_1 = -3$, then $Y_1 = -4/7 < 0$, but the gain is $V_1 - V_0 = 4 > 0$.

Rates of return are often expressed by time series: let time $t-1$ be the starting point of period t, then the return of period t, $Y(t-1,t)$, is written by Y_t for brevity.

Remark: In finance, the word return may refer to profit or rate of return. In financial economics, gross rate of return, V_t/V_s, is often called return for short.

1. A synonym of profit, the gain or loss on an investment. A return over the period from time s to time t is the value change $V(t) - V(s)$.
2. Return is also used to refer to holding period return, defined in Eq (4.11), generally expressed as a percentage.

A: Interest Rate and Return

In a setting with simple interest at rate r, the interest rate is equal to the return over one year

$$Y_1 = r$$

In a setting with the rate r per annum compounded m times per annum, by Eq (4.2)

$$Y(s,t) = \left(1 + \frac{r}{m}\right)^{(t-s)m} - 1$$

[1]A real-world short sale of a common stock is in effect a "long" position in the stock with a margin account, but one where the short-seller arranges to get the negative of the return on the usual type of purchase.

In particular, for a compounding period

$$Y(0,1/m) = \frac{1}{m} \cdot r$$

The return on a deposit subject to periodic compounding is not additive. For example, take $m = 1$. Then

$$Y_1 = Y_2 = r$$

however

$$Y(0,2) = (1+r)^2 - 1 \neq 2r = Y_1 + Y_2$$

Remark: In the circumstance of uncertainty, $Y(s,t)$ is a random variable at time s, and is realized at time t. However, interest rate is known at time s, the beginning of investment.

- As a general rule, interest rates will usually refer to a period of one year, facilitating the comparison between different investments. Annualized interest rates are independent of their actual duration.
- By contrast, the return reflects both the interest rate and the length of time the investment is held.

Similar to nominal annual rate, *annualized rate of return* is defined by $Y_a = Y(s,t)/(t-s)$. Which simply scales holding period return by the number of periods in one year.

B: Log Return

With continuous compounding, we define the *log return* (logarithmic return, also called *continuously compounded return*, *log price difference*) by

$$Y(s,t) = \ln\left(\frac{V(t)}{V(s)}\right) = \ln(V_t) - \ln(V_s)$$

Note that the log return is additive

$$Y(s,t) + Y(t,u) = Y(s,u) \qquad s < t < u$$

The *annualized log return* is computed by

$$r = \frac{Y(s,t)}{t-s} = \frac{\ln(V_t) - \ln(V_s)}{t-s}$$

Remark: average log return depends merely on the starting and ending time point.

§ 4.2 Term Structure

A graph showing representations of various interest rates for progressively longer terms is called the *term structure*. The term structure of interest rates, also known as the *yield curve*, is commonly designed to be flat in valuation of cash flow streams in residential mortgage loans. Although a world of deterministic interest rate is unreal and uninteresting, even with time-varying interest rates, however, spot rates and forwards rates are better understood in this simple settings.

4.2.1 Constant Interest Rate

Suppose that one can both borrow and loan money at a nominal rate r_p per period (h years), simply compounded. For clarity, let the beginning of the nth period be time $n-1$ (more exactly, $t=(n-1)h$) and the end point be time n.

A: Present Value

When valuing a stream of cash flows extending over a number of periods, the interest rate is often quoted by periodic rate. For cash flow C_n at time n (at the end of period n), the *present value*

$$P = \wp_{0,n}(C_n) = \frac{C_n}{(1+r_p)^n}$$

For a stream of cash flows $\{C_n : n = 1, 2, \cdots\}$

$$P = \sum_n \wp_{0,n}(C_n) = \sum_n \frac{C_n}{(1+r_p)^n}$$

We can convert the simple compounding to continuous compounding, let r be the periodic rate compounded continuously, by $1+r_p = e^r$, we have

$$r = \ln(1+r_p)$$

With continuous compounding, for cash flow C_n at time n

$$P = \wp_{0,n}(C_n) = C_n e^{-(r/h)\cdot nh} = C_n e^{-rn}$$

For a stream of cash flows $\{C_n : n = 1, 2, \cdots\}$

$$P = \sum_n \wp_{0,n}(C_n) = \sum_n C_n e^{-rn}$$

An annuity is a cash flow stream that pays a fixed sum each period for a specified number of periods. If the number of cash flows is N, let $C_n = C$, $n = 1, 2, \cdots, N$, then

$$P = \frac{1-e^{-rN}}{e^r - 1} C = \mathrm{PV}(r, N)\cdot C \tag{4.12}$$

with *present value factor*

$$\mathrm{PV}(r, N) = \frac{1-e^{-rN}}{e^r - 1}$$

equivalently, in periodic rate r_p

$$\mathrm{PV}(r_p, N) = \frac{1-(1+r_p)^{-N}}{r_p}$$

The present value (PV) formula has four variables, each of which can be solved given the other three.

The equal-payment house mortgage or installment credit agreement are common examples of annuities. Where a series of equal payments or receipts occurs at evenly spaced intervals.

B: Mortgage Loan

A residential mortgage loan is secured by a lien on the real property, a house or other residential property in which the borrowers will live, the borrowers repay it over a specified period of time.

Example 4.2.1: Suppose that one takes a mortgage loan for the amount L that is to be paid back over N months with equal payments of A at the end of each month. The interest rate for the loan is r per month, compounded monthly.

(a) In terms of L, N, and r, what is the value of A?

(b) After payment has been made at the end of month i, how much additional loan principal remains?

(c) How much of the payment during month i is for interest and how much is for principal reduction? (This is important because some contracts allow for the loan to be paid back early and because the interest part of the payment is tax-deductible.)

(a) The present value of the N monthly payments equals the loan amount L, thus

$$A = \frac{L}{\mathrm{PV}(r, N)} = \frac{rL}{1-(1+r)^{-N}} \tag{4.13}$$

(b) Let L_i denote the remaining amount of principal owed after the payment at the end of month i (time $i = 1, 2, \cdots, N$). It follows that

$$(1+r)L_i = A + L_{i+1}$$

with $L_N = 0$. Let $y = 1/(1+r)$, we obtain

$$\begin{aligned} L_i &= yL_{i+1} + yA = y(yL_{i+2} + yA) + yA \\ &= yL_{i+2} + (y + y^2)A = \cdots \\ &= yL_N + (y + y^2 + \cdots + y^{N-i})A \\ &= y\frac{1-y^{N-i}}{1-y}A = \frac{1-(1+r)^{i-N}}{r}A \end{aligned}$$

As a check, at the beginning of first month $L_0 = L$.

(c) Let I_i and P_i denote the amounts of the payment at the end of month i that are for interest and for principal reduction, respectively. Then

$$\begin{aligned} I_i &= rL_{i-1} = (1-(1+r)^{i-1-N})A \\ P_i &= A - I_i = (1+r)^{i-1-N}A \end{aligned}$$

As a check, we see that $\sum_{i=1}^{N} P_i = \mathrm{PV}(r, N) \cdot A = L$

Remark: the interest rate in Eq (4.13) is r per month, compounded monthly. If the interest is quoted per year compounded monthly, it should be converted to per month compounded monthly.

4.2.2 The Yield Curve

The yield curve shows the relation between the (level of) interest rate (or cost of borrowing) and the time to maturity, known as the "term", of the debt for a given borrower in a given currency. The yield curve is one of the three[2] "curves" used routinely by market professionals. Who analyze bonds and related securities, to understand conditions in financial markets and to seek trading opportunities.

A: Shapes of the Yield Curve

Let $D_{t,T} = D(t,T)$ be the price of a discount bond (zero-coupon bond, or T-bond) paying off \$1 at time T observed at time $t < T$. There are no intermediate payments, thus

$$D_{t,T} = \wp_{t,T}(1)$$

In particular, for any time t

$$D(t,t) = 1$$

Assume that at time t there exist zero-coupon bonds with maturities ranging from t to the longest maturity available in the market.

Let $R_{t,T} = R(t,T)$ be the *spot rate* for $T - t$ year observed at time $t < T$, then

$$V_T = V_t e^{R_{t,T}(T-t)}$$

Say, $R_{t,T}$ is the nominal annual rate of interest (compounded continuously) earned on an investment that starts at time t and lasts for $T - t$ years. Thus, $\wp_{t,T}(1)$ is set by $R_{t,T}$ and

$$D_{t,T} = \wp_{t,T}(1) = e^{-R_{t,T}(T-t)} \tag{4.14}$$

Remark: We see that $R_{t,T}$ is the $(T-t)$-year zero-coupon interest rate (or simply *zero rate*), or the yield to maturity (YTM) of $D_{t,T}$ at time t.

The spectrum of yields $\{R(t,T) : T \geqslant t\}$ is called the yield curve. At time t, investing for a period of time h gives a yield $Y_t(h) = R(t, t+h)$. This function $Y_t(h)$ is called the *yield curve* at time t, and it is often, but not always, an increasing function of h. The yield curve function $Y_t(h)$ is actually only known with certainty for a few specific maturity dates, while the other maturities are calculated by interpolation.

Remark: The usual representation of yield curve is

$$Y_t(h) = e^{R_{t,t+h}} - 1$$

for professionals, $Y_t(h)$ is written in effective rate. However, we will use textbook representation of $Y_t(h) = R(t, t+h)$ for convenience, where $Y_t(h)$ is a continuous rate.

The yield curve plays a central role in an economy. The term structure reflects expectations of market participants about future changes in interest rates and their assessment of monetary policy conditions. Economists use the curves to understand economic conditions. In most situations, it is an upward sloping curve, yields increase in line with maturity. One basic explanation for this phenomenon is that interest

[2]These are the yield curve, the discount curve, and the credit-spread curve.

rates for longer-term loans are associated with greater risk, and a risk premium as a compensation is needed by the market. However, a positively sloped yield curve has not always been the norm. Occasionally, long-term yields may fall below short-term yields, creating an "inverted" yield curve that is generally regarded as a harbinger of recession. Aside from the normal and inverted yield curve, the shapes of the yield curve may also be steep, flat or humped.

B: Short Rate

From Equation (4.14)

$$R_{t,T} = R(t,T) = -\frac{\ln(D(t,T))}{T-t}$$

The spot rate $R(t,T)$ is observed at time t, and apply for the period of time from t to T. The *instantaneous spot rate* is called the *short rate*, which is computed by

$$r_t = \lim_{h\to 0} R(t,t+h) \equiv R(t,t) = -\left.\frac{\partial \ln(D(t,T))}{\partial T}\right|_{T=t} \tag{4.15}$$

The short rate, r_t, then, is the interest rate (continuously compounded, annualized) at which an investor can borrow money for an infinitesimally short period of time from time t. For this reason, sometimes, the $R_{t,T}$ is called *long rate*.

Remark: Short rate is not the short term interest rate. Short term interest rates are interest rates on loans or debts having maturities of less than one year. Often called money market rates. The overnight rate, only available to the most creditworthy institutions, is the shortest term and lowest interest rate.

The risk-free growing bond $B(t)$ is a theoretical bond. Inspired by Eq (4.4), the process of $B(t)$ is defined as

$$\frac{\mathrm{d}B_t}{\mathrm{d}t} = r_t B_t \tag{4.16}$$

Solving out this differential equation[3] with $B_0 = 1$

$$B(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right) \tag{4.17}$$

When $r(t,w)$ is random, the integral[4] is computed pathwise for each market scenario $w \in \Omega$.

Remark: In an infinitesimal time interval h (a finitely short time interval $h > 0$ that is approaching zero. We are examining at a small scale that is a mathematical abstract), we write $x \sim y$ if x and y are equal asymptotically in the sense that $\lim_{h\to 0} \frac{x}{y} = 1$

- Long rate is equal to short rate asymptotically, $r_t \sim R(t,t+h)$
- The rate of continuous compounding interest is asymptotically equal to the rate of simple compounding

$$e^{r_t h} \sim 1 + r_t h$$

[3]The ODE is separable in B_t, for $\mathrm{d}\ln(B_t) = \frac{\mathrm{d}B_t}{B_t} = r_t \mathrm{d}t$, by direct integration, $\int_0^t \mathrm{d}\ln(B_s) = \int_0^t r_s\,\mathrm{d}s$, which yields $\ln(B_t) - \ln(B_0) = \int_0^t r_s\,\mathrm{d}s$, and thus Equation (4.17).

[4]When short rate r_t is model by SDE (stochastic differential equation), we need the Itô calculus, for the Riemann integral fails pathwise because of the unbounded variation.

- In a world of deterministic interest rate (or a path in random world)
$$B(t+h) = B(t)e^{R(t,t+h)h} \sim B(t)e^{r_t h} \sim B(t)\cdot(1+r_t h)$$
and
$$\frac{\mathrm{d}B_t}{\mathrm{d}t} = \lim_{h\to 0}\frac{B(t+h)-B(t)}{h} = B_t r_t$$

C: Forward Rates

We need to define a continuously compounded *forward rate* $f(t,S,T)$. This concept represents the interest rate on a loan that begins at time S and matures at time $T > S$. The rate is contracted at time t, although cash transactions will take place at future dates S and T. At forward rate $f(t,S,T)$, if one dollar is deposited at time S, the money withdrawn at time T will be $e^{f(t,S,T)(T-S)}$.

Example 4.2.2: Supposed that a buyer enters into a forward contract at time t to deposit one dollar at future time S and withdraw $F = D(t,S)/D(t,T)$ dollars at terminal time T. Show that the value of this contract at time t will be zero.

This forward contract is a portfolio of one short position in S-bond and F long positions in T-bond, thus
$$F_t = \wp_{t,S}(-1) + \wp_{t,T}(F) = -1\cdot D(t,S) + (D(t,S)/D(t,T))\cdot D(t,T) = 0$$
the value of this forward contract at time t is zero. Note that $F_t \neq \wp_{t,T}(F)$, because there is an addition of one dollar at time S to close out the short position.

In the example 4.2.2 above, let $f(t,S,T)$ be the forward rate, then
$$1\cdot e^{f(t,S,T)(T-S)} = F = D(t,S)/D(t,T)$$
which yields
$$f(t,S,T) = \frac{\ln(D_{t,S}) - \ln(D_{t,T})}{T-S} = \frac{(T-t)R_{t,T} - (S-t)R_{t,S}}{T-S} \tag{4.18}$$
We see that forward rates are interest rates that can be locked[5] in today at zero cost for an investment in a future time period. To avoid arbitrage, one dollar invested during the future time interval $[S,T]$ contracted at time t for a sure return must yield an interest rate at $f(t,S,T)$. Thus, the forward rate $f(t,S,T)$ defined in Eq (4.18) is consistent with the current term structure.

The *instantaneous forward rate* is defined by
$$f(t,T) = \lim_{h\to 0} f(t,T,T+h) \equiv f(t,T,T)$$
Therefore
$$f(t,T) = -\frac{\partial \ln(D(t,T))}{\partial T} = R(t,T) + (T-t)\frac{\partial}{\partial T}R(t,T) \tag{4.19}$$

[5]At time t, we take one short position in S-bond and $D(t,S)/D(t,T)$ long position in T-bond, this is a zero investment shown in Example 4.2.2. At time S, the S-bond matures and we have to pay out one dollar. At time T, the T-bond matures and we receive $D(t,S)/D(t,T)$ dollars. The net effect of this operation is a forward investment at the fixed rate of $f(t,S,T)$, such that one dollar at time S yields $e^{f(t,S,T)(T-S)} = D(t,S)/D(t,T)$ dollars at T with certainty.

and

$$D(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right) \tag{4.20}$$

$$R(t,T) = \frac{1}{T-t}\int_t^T f(t,u)\,\mathrm{d}u \tag{4.21}$$

Note that forward rate $f(t,S,T)$ has three time parameters, we have

$$R(t,T) = f(t,t,T)$$

$$f(t,T) = f(t,T,T)$$

and

$$r_t = \lim_{h\to 0} f(t,t,t+h) \equiv f(t,t,t) = R(t,t) = f(t,t)$$

thus, forward rate $f(t,S,T)$ is a key and a bridge among all kinds of interest rates.

4.2.3 Time-varying Interest Rate

If the short rates are given as a deterministic function of time, by Eq (4.17)

$$B_T = \exp\left(\int_0^T r(s)\,\mathrm{d}s\right)$$

is known at time t, we have

$$B_t = \wp_{t,T}(B_T) = B_T\wp_{t,T}(1) = B_T D_{t,T}$$

thus

$$D(t,T) = \frac{B_t}{B_T} \tag{4.22}$$

When short rate $r(t)$ is a deterministic function of time, then interest rates are not random, but allow some shape of yield curve, we say that the interest rates are time-varying but deterministic.

A: Future Spot Rate

$R_{t,T}$ is a spot rate seen at time t, moreover, it is a *future spot rate* imagined at time $s < t$. Since the future state is known with certainty, we have the following properties:

Proposition 4.1: When interest rates are time-varying but deterministic, we have the following statements, and they are equivalent

(a) The future spot rate is set by forward rate

$$R(S,T) = f(t,S,T) \qquad t \leqslant S \leqslant T \tag{4.23}$$

(b) The future instantaneous spot rate must be the instantaneous forward rate

$$r_u = f(t,u) \qquad t \leqslant u \tag{4.24}$$

(c) The spot rate $R_{t,T}$ is an average of short rate r_t over a period from time t to T

$$R(t,T) = \frac{1}{T-t}\int_t^T r(u)\,\mathrm{d}u \qquad t \leqslant T \tag{4.25}$$

The proof of Proposition 4.1 is left as Exercise 4.13.

In a time-varying but deterministic setting, Equation (4.23) states that the forward rate $f(t, S, T)$ for period from time S to T observed at time t, must be the future spot rate $R(S, T)$ at time S with certainty. For the instantaneous forward rate, we have the following relationship

$$f(u, u) = r_u = R(u, u) = f(t, u, u) = f(t, u) \qquad t \leqslant u$$

A non-stochastic world is not very interesting, the future is totally known in advance. For example, $R(S, T)$ is known at time $t < S$.

B: Future Yield Curve

When interest rates are time-varying but deterministic, the future yield curve is also known given the current yield curve.

Proposition 4.2: When interest rates are time-varying but deterministic, the yield curve at time $u > t$, $Y_u(h)$ is uniquely implied by $Y_t(h)$, with

$$Y_u(h) = \frac{u - t + h}{h} Y_t(u - t + h) - \frac{u - t}{h} Y_t(u - t) \tag{4.26}$$

the evolution of the yield curve is completely determined by the yield cure observed at time t.

The yield curve has usually been "normal" meaning that yields rise as maturity lengthens. However, if the world is deterministic, a normal yield curve means the future spot rates will be rising (Exercise 4.16). If the yield curve is flat, $Y_0(h) = r$, such that spot rates are constant, then the future yield curve is still flat, and the future spot rates remain unchanged, $R(t, T) = r_t = r$ constant for any t and T.

Example 4.2.3: Given

$$Y_0(h) = a + \frac{b}{h} \ln(1 + h)$$

Please: Find $R(t, T)$ and $D(t, T)$; Compute $r(t)$ by $R(t, t)$; Verify Eq (4.15) and (4.25).

By Eq (4.26)

$$R(t, T) = Y_t(T - t) = \frac{T}{T - t} Y_0(T) - \frac{t}{T - t} Y_0(t) = \frac{b}{T - t} \ln\left(\frac{T + 1}{t + 1}\right) + a$$

By Eq (4.14)

$$D(t, T) = e^{-R(t,T)(T-t)} = \left(\frac{t + 1}{T + 1}\right)^b e^{-a(T-t)}$$

The short rate r_t is

$$r_t = R(t, t) = a + b \lim_{T \to t} \frac{\ln\left(\frac{T+1}{t+1}\right)}{T - t} = a + b \lim_{T \to t} \frac{\frac{t+1}{T+1}\frac{1}{t+1}}{1} = a + \frac{b}{t + 1}$$

For

$$\frac{\partial \ln\left(D\left(t, T\right)\right)}{\partial T} = \frac{\partial}{\partial T}\left(b \ln\left(\frac{t + 1}{T + 1}\right) - a(T - t)\right) = -\left(a + \frac{b}{T + 1}\right)$$

thus

$$-\left.\frac{\partial \ln\left(D\left(t, T\right)\right)}{\partial T}\right|_{T=t} = a + \frac{b}{t + 1} = r_t$$

which verifies Eq (4.15). And

$$\int_t^T r(u)\,\mathrm{d}u = \int_t^T a + \frac{b}{u+1}\,\mathrm{d}u = au + b\ln(u+1)\Big|_t^T$$
$$= a(T-t) + b\ln\left(\frac{T+1}{t+1}\right) = (T-t)\cdot R(t,T)$$

which verifies Eq (4.25).

C: Information Contained in Interest Rates

Proposition 4.3: In a world of deterministic interest rates, the following statements are equivalent

(a) The short rate $r(t)$ is a deterministic function of time t

(b) The price process of risk-free bond $B(t)$ is a deterministic function of time t

(c) The price process of discount bond $D(t,T)$ is a deterministic function of time t and T

(d) The spot rate $R(t,T)$ is a deterministic function of time t and T

(e) The term structure $Y_t(h)$ is a deterministic function of time t and term h

(f) The instantaneous forward rate $f(t,T)$ is a deterministic function of time t and T.

(g) The forward rate $f(t,S,T)$ is a deterministic function of time t, S and T.

When the world is non-stochastic, given anyone of $r(t)$, $B(t)$, $D(t,T)$, $R(t,T)$, $Y_t(h)$, $f(t,T)$ and $f(t,S,T)$, the others will be also known. The forward rate $f(t,S,T)$ is a function of three variables, spot rate $r(t)$ is a function on only one variable. It seems that $f(t,S,T)$ contains more information than $r(t)$, however, they contain the same information for they are fully determined by each other.

§ 4.3 Stochastic Interest Rates

What is the meaning of random risk-free interest rates? Why the inherent uncertainty can be risk-free. When we are talking about randomness of risk-free rate, we are talking about the future spot rates, $R(t,T)$ or r_t in future time t. At time t, both $R(t,T)$ and r_t are known, at time $s<t$, $R(t,T)$ and r_t are not known, they are depicted by random variables.

We know the value of $D(t,T)$ at time t and T exactly, at other time points, $D(t,T)$ is random. If we hold ZCB to maturity T, it is riskless.

4.3.1 Future Spot Rate

When r_t is model by an Itô process (in risk-neutral world), Eq (4.17) should be interpret as stochastic calculus.

If the risk-free rates are random

$$D\left(t,T\right)=B_t\,\mathrm{E}_t^Q\left(\frac{1}{B_T}\right)=\mathrm{E}_t^Q\left(\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right)\right)$$

and

$$R(t,T)\neq\frac{1}{T-t}\int_t^T r\left(u\right)\,\mathrm{d}u$$

For the LHS is a known value at time t, but RHS is a random variable, by the definition of spot rate, there is

$$R(t,T)=-\frac{1}{T-t}\ln\left(\mathrm{E}_t^Q\left(\exp\left(-\int_t^T r(u)\,\mathrm{d}u\right)\right)\right)$$

Remark: since $B(t)$, $D(t,T)$ and $R(t,T)$ are fully determined by r_t, similar to Proposition 4.3, given anyone of $r(t)$, $B(t)$, $D(t,T)$, $R(t,T)$, $Y_t(h)$, $f(t,T)$ and $f(t,S,T)$, the others will be also known.

$$B\left(t\right)=\exp\left(\int_0^t r(s)\,\mathrm{d}s\right)$$

$$\frac{\mathrm{d}B_t}{\mathrm{d}t}=r_tB_t\implies r_t=\frac{1}{B_t}\frac{\mathrm{d}B_t}{\mathrm{d}t}$$

$$D\left(t,T\right)=B_t\,\mathrm{E}_t^Q\left(\frac{1}{B_T}\right)$$

$$D_{t,T}=\wp_{t,T}(1)=e^{-R_{t,T}(T-t)}\implies R_{t,T}=-\frac{\ln(D(t,T))}{T-t}$$

$$f\left(t,S,T\right)=\frac{\ln\left(D_{t,S}\right)-\ln\left(D_{t,T}\right)}{T-S}=\frac{(T-t)R_{t,T}-(S-t)R_{t,S}}{T-S}$$

A: Expectations Hypothesis

The *expectations hypothesis* of the term structure of interest rates conjectures that long-term interest rates should reflect expected future short-term interest rates. Given long-term rates, forward rates are purely determined by current and expected future short-term rates, thus $f(t,S,T)=\mathrm{E}_t(R(S,T))$. However, the future spot rates are determined by future economy, not fully determined by current economy and forward rates.

It seems that the expectations hypothesis is not a valid conjecture, for

$$f(t,S,T) \neq \mathrm{E}_t(R(S,T))$$

and

$$f(t,S,T) \neq \mathrm{E}_t^Q(R(S,T))$$

We have some knowledge (thought, thinking, opinion) of a world with certainty. However, when these judgments are extended to a world of uncertainty, there may not work.

The value of FRA in Example 4.2.2 at time S is

$$F_S = \wp_{S,T}(F_T) + \wp_{S,S}(-1) = e^{f(t,S,T)(T-S)} D_{S,T} - 1$$

If $F_S = 0$, then $f(t,S,T) = R(S,T)$. On the other hand, if $f(t,S,T) = R(S,T)$, $F_S = 0$. However, at time $t < S$, either $R(S,T)$ or F_S is not known. If $\mathrm{E}_t(F_S) = 0$, we have

$$\mathrm{E}_t(e^{-R(S,T)(T-S)}) = \mathrm{E}_t(D_{S,T}) = e^{-f(t,S,T)(T-S)}$$

not the straightforward expression $f(t,S,T) = \mathrm{E}_t(R(S,T))$.

Let us now switch into the setting of simple compounding (compounded per period to which $R_{S,T}$ applies), if $\mathrm{E}_t(F_S) = 0$, we arrive at

$$\mathrm{E}_t\left(\frac{1}{1+R(S,T)(T-S)}\right) = \frac{1}{1+f(t,S,T)(T-S)}$$

there is no simple result as $f(t,S,T) = \mathrm{E}_t(R(S,T))$. Let

$$G = \Psi / D_{t,T} > 0$$

then $\mathrm{E}_t(G) = 1$, we can define an equivalent probability measure O by $\mathrm{d}O = G\,\mathrm{d}P$. Given payoff X_T, if $X_t = \wp_{t,T}(X_T)$, then under probability measure O, $X_t/D_{t,T}$ is a martingale under measure O (Equation 3.22)

$$\frac{X_t}{D_{t,T}} = \mathrm{E}_t^O\left(\frac{X_T}{D_{T,T}}\right) = \mathrm{E}_t^O\left(\frac{X_S}{D_{S,T}}\right) \qquad t \leqslant S < T \tag{4.27}$$

Probability measure O is called T-forward measure due to the following proposition.

Proposition 4.4: Under probability measure O, the forward rate is the expectation of future spot rate (simply compounded)

$$f(t,S,T) = \mathrm{E}_t^O(R(S,T))$$

Proof. Let (self-financing process)

$$X_t = \begin{cases} D(t,S) & t \leqslant S \\ B_t/B_S & S < t \leqslant T \end{cases}$$

by Eq (4.27), there is

$$\mathrm{E}_t^O\left(\frac{D(S,S)}{D(S,T)}\right) = \mathrm{E}_t^O\left(\frac{X_S}{D_{S,T}}\right) = \frac{X_t}{D_{t,T}} = \frac{D(t,S)}{D(t,T)}$$

thus (by $R(S,T) = f(S,S,T)$ and Exercise 4.19)

$$\mathrm{E}_t^O(R(S,T)) = \mathrm{E}_t^O(f(S,S,T)) = \frac{1}{T-S}\left(\mathrm{E}_t^O\left(\frac{D(S,S)}{D(S,T)}\right) - 1\right)$$
$$= \frac{1}{T-S}\left(\frac{D(t,S)}{D(t,T)} - 1\right) = f(t,S,T)$$

□

Considering the portfolio

$$X_t = \begin{cases} \frac{D(t,S)-D(t,T)}{T-S} & t \leqslant S \\ \frac{(B_t/B_S)-D(t,T)}{T-S} & S < t \leqslant T \end{cases}$$

by simple compounding, there is (Exercise 4.19)

$$f(t,S,T) = \frac{X_t}{D(t,T)} \qquad t \leqslant S$$

Eq (4.27) shows that $f(t,S,T)$ is a martingale under measure O, thus

$$\mathrm{E}_t^O(R(S,T)) = \mathrm{E}_t^O(f(S,S,T)) = f(t,S,T)$$

Remark: $f(t,S,T)$ is not a price process. Only if the numeraire is changed to $D(t,T)$, $f(t,S,T)$ is a price process of trading asset $X_t = \frac{1}{T-S}(D(t,S) - D(t,T))$ at $t \leqslant S$.

Remark: In a world of deterministic interest rate, $G = \Psi/D_{t,T} = \Psi B_T/B_t$, the T-forward measure is risk-neutral measure.