CHAPTER 1

Introduction

In financial market, every asset has a price. However, most often than not, the price are not equal to the expectation of its payoff. It is widely accepted that the pricing function is linear. Thus, we have a first look on the linearity of pricing function in forms of risk-neutral probability, state price and stochastic discount factor. To begin our analysis of pure finance, we explain the perfect market assumptions and propose a group of postulates on asset prices. No-arbitrage principle is a cornerstone of the theory of asset pricing, absence of arbitrage is more primitive than equilibrium. Given the mathematical definition of an arbitrage opportunity, we show that absence of arbitrage is equivalent to the existence of a positive linear pricing function.

§ 1.1 Motivating Examples

The risk-neutral probability, state price, and stochastic discount factor are fundamental concepts for modern finance. However, these concepts are not easily understood. For a first grasp of these definitions, we enter a fair play with elementary probability and linear algebra. Linear pricing function is the vital presupposition on financial market, therefore, we present more examples on it, for a better understanding of its application and arbitrage argument.

1.1.1 Fair Play

Example 1.1.1 (fair play): Let's play a game X: you pay \$1. I flip a fair coin. If it is heads, you get \$3; if it is tails, you get nothing.

$$X_0 = 1 \quad \longrightarrow \quad X = \begin{cases} x_H = 3 \\ x_L = 0 \end{cases}$$

However, you like to play big, propose a game Y that is more exciting: where if it is heads, you get \$20; if it is tails, you get \$2.

$$Y_0 = ? \quad \longleftarrow \quad Y = \begin{cases} y_H = 20 \\ y_L = 2 \end{cases}$$

What is the fair price of game Y? By expectation? Let's assume both of us have struck oil, if one of us would like to play a game the other one will serve as a dealer (banker).

A fair coin means that both heads and tails occur with equal chance, with a probability of 0.5. If the price of game Y is computed by expectation, so it will be

$$Y_0 = \mathrm{E}(Y) = 0.5 \cdot 20 + 0.5 \cdot 2 = 11$$

However, that will not be a fair play. If you are willing to accept the price of game Y to be \$11, I will play 6 on game X (multiplied by 6, or place 6 bets, not repeatedly play it 6 times) and invite you to play one game Y against me (I am a dealer on Y and a player on X). Then, before the coin is tossed, I collect $1 \cdot \$11 - 6 \cdot \$1 = \$5$. And no matter what the outcomes of toss are, head or tail, I should pay you \$2: If it is heads, $1 \cdot \$20 - 6 \cdot \$3 = \$2$; and if it is tail, $1 \cdot \$2 - 6 \cdot \$0 = \$2$. Obviously, I earn \$3 from this scheme, it is an arbitrage opportunity, a free lunch!

Remark: Although it has been used by actuaries for centuries, the instinctive method of calculating prices via expectation is not directly applicable. It already fails for the game X, $\mathrm{E}(X) = 3/2 \neq X_0$, and hence there is no reason why the price of game Y should be given by its expectation. Besides, if the interest rate is positive, pricing by expectation is obviously incorrect because of the lacking of discount.

A: Financial Engineering

Now we state the game in the viewpoint of finance: there is a *risky asset* X traded in the market at price X_0 with *payoff* as a random variable X. A client request a new asset with payoff Y, what is the fair price of Y?

Remark: Multiple meanings of X, asset X and payoff X

- Abuse of notation: an asset is identified by its payoff, thus, we use X for both asset and its payoff.
- Asset's payoff: it is just another fancy name for the future value of an asset, the future income by selling it. For the ticket of game X, the payoff X is the value of the ticket when the outcome of the coin flip is known.

As a financial engineer, you are asked to create asset Y, and set a fair price, the right price that is not too high or too low to avoid arbitrage opportunities. We should first make clear what are the assets available in the market? There is an asset X, and a risk-free asset. Where is the risk-free asset? You can keep the money in your pocket —— A *risk-free asset* with constant interest rate $r = 0$. We denote this risk-free asset as bond B, with price $B_0 = 1$ and a deterministic payoff $B = 1$ (irrelevant to the outcome of the coin flip).

To create a new financial instrument, the most widely used method is *replication*: we design a portfolio of traded assets, which has the same payoff as needed. For example, to create asset Y, we form a portfolio Z by

$$\text{Buy: 6 asset } X \text{ and 2 bond } B$$

When a portfolio Z is a *long position* of 6 asset X and 2 bond B, we mean that *portfolio* Z is a package of 6 shares of asset X and 2 shares of bond B. If it is heads

$$z_H = 6 \cdot x_H + 2 \cdot B = 20 = y_H$$

if it is tails, $z_L = 6 \cdot x_L + 2 \cdot B = 2 = y_L$. Therefore

$$Z = 6 \cdot X + 2 \cdot B = Y$$

which is a replication of asset Y. Once asset Y is replicated, it goes without saying that the price of asset Y follows automatically

$$Y_0 = Z_0 = 6 \cdot X_0 + 2 \cdot B_0 = 6 \cdot 1 + 2 \cdot 1 = 8$$

the first equality, $Y_0 = Z_0$, follows from the fact that Y and Z have the same payoff, they are indiscrimination as financial assets, thus they must have the same price. And the second equality

$$Z_0 = 6 \cdot X_0 + 2 \cdot B_0 \tag{1.1}$$

cast no doubt: we buy 6 shares of asset X and 2 shares of bond B to form the portfolio Z with a cost of $6 \cdot X_0 + 2 \cdot B_0$. The price of a package of goods (assets), is equal to the total price of its items. Otherwise, there will arise an arbitrage opportunity: if someone would like to accept Z with price $Z_0 = 11 > 8$, we apply the following strategy (buy low sell high)

$$\text{Buy: 6 asset } X \text{ and 2 bond } B$$

$$\text{Sell: one asset } Z$$

say, sell the package, and buy 6 shares of X and 2 shares of B. Then we are happy to pocket the profit of \$3, and free to walk away with no obligation.

B: Risk-Neutral Probability

Financial engineers live in two world, P world and Q world. The P world is the physical world, where the probability is 0.5 for a fair coin to land on heads. Q world is somewhat mysteriously called risk-neutral world. Yes, it is a suppositional world, where a set of different probabilities from real world is used. In Q world, investors are neither risk averse nor risk seeking, they are indifferent between choices with equal expected payoffs even if one choice is riskier.

Why do we need the Q world? Let's enter the Q world following the pricing of game Y. Suppose that we are able to long a shares X and b shares B to replicate Y, then $Y_0 = aX_0 + b$, and

$$Y = aX + bB$$

Which means when it is heads

$$y_H = ax_H + b$$

and when it is tails

$$y_L = ax_L + b$$

Thus, to find out the unknowns a and b, we solve the following system of linear equations

$$\begin{aligned} x_H \cdot a + b &= y_H \\ x_L \cdot a + b &= y_L \end{aligned} \tag{1.2}$$

However, we try a shortcut: Let's multiply q (to be determined) to the first equation and $1-q$ to the second equation, and sum them up

$$\begin{aligned} qy_H + (1-q)y_L &= q(x_H \cdot a + b) + (1-q)(x_L \cdot a + b) \\ &= a[qx_H + (1-q)x_L] + b \\ &= aX_0 \cdot [qx_H + (1-q)x_L]/X_0 + b = aX_0 + b \end{aligned}$$

by setting

$$[qx_H + (1-q)x_L]/X_0 = 1$$

or

$$q = \frac{X_0 - x_L}{x_H - x_L} \tag{1.3}$$

We see that q is fully determined now. Furthermore, we find the price straightforward and elegantly

$$Y_0 = aX_0 + b = qy_H + (1-q)y_L$$

For $0 < q < 1$ (since $x_L < X_0 < x_H$), consider the following transformation of the original probabilities associated with tossing the coin

$$\mathrm{P}(H) = \frac{1}{2} \to \mathrm{Q}(H) = q$$
$$\mathrm{P}(L) = \frac{1}{2} \to \mathrm{Q}(L) = 1 - q$$

where we interpret q from Eq (1.3) as the probability of heads. Let $\mathrm{E}^Q(\cdot)$ be the expectation under the new probability settings $\{\mathrm{Q}(H), \mathrm{Q}(L)\}$, we have

$$Y_0 = \mathrm{E}^Q(Y) \tag{1.4}$$

Which states that the price is the expected value of its payoff under this new probability settings. In finance, this set of new probabilities is called *risk-neutral*[1] *probability measure*.

Given X_0, x_H and x_L in Example 1.1.1, by Eq (1.3)

$$q = \frac{X_0 - x_L}{x_H - x_L} = \frac{1}{3}$$

thus the price of Y is

$$Y_0 = \mathrm{E}^Q(Y) = y_H \mathrm{Q}(H) + y_L \mathrm{Q}(L) = 20 \cdot \frac{1}{3} + 2 \cdot \left(1 - \frac{1}{3}\right) = 8$$

Remarks: The Q world is a mathematically convenient simplification

- Observing on the equation (1.4), the actual probabilities (P world) is irrelevant (One should not be concerned, as Exercise 1.2).
- By taking logarithm, we can convert the multiplications to additions. By employing risk-neutral probabilities, we transform solving a system of linear equations into taking expectations.
- The q in Eq (1.3) acts like an auxiliary line in solving geometry problems, with the aid of q, we find the price immediately without the difficulties of solving a system of linear equations.

C: Risk-Neutral Pricing

The system (1.2) is often written in a standard matrix form

$$\mathbf{X}'\mathbf{h} = \mathbf{y} \tag{1.5}$$

where

$$\mathbf{X} = \begin{bmatrix} x_H & x_L \\ 1 & 1 \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_H \\ y_L \end{bmatrix}$$

for asset X is risky, with $x_H > x_L$, matrix $\mathbf{X}$ is full rank, and thus system (1.2) has a unique solution

$$\mathbf{h} = (\mathbf{X}^{-1})'\mathbf{y}$$

[1]Using probability $q = \frac{1}{3}$ in Eq (1.3), we have $Y_0 = \mathrm{E}^Q(Y)$ and $X_0 = \mathrm{E}^Q(X)$, consequently, the expected returns of asset X and Y equal the risk-free rate $r = 0$. We find that investors do not require a higher or lower (negative risk) expected returns to compensate for the riskiness of an investment: accordingly, investors are risk-neutral in Q world.

for any $\mathbf{y}$. To allow $\mathbf{y}$ to take any value is equivalent to allow asset Y to be any payoff (random variable). Let price vector of primitive assets be

$$\mathbf{p} = \begin{bmatrix} X_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} X_0 \\ 1 \end{bmatrix}$$

we see that

$$\mathbf{q} = \begin{bmatrix} q \\ 1-q \end{bmatrix} = \mathbf{X}^{-1}\mathbf{p} \tag{1.6}$$

Clearly, q does not depend on either real world probability $\mathrm{P}(H)$ or asset Y. Reading as a probability, q is uniquely determined by the market settings, and

$$Y_0 = aX_0 + b = \mathbf{h}'\mathbf{p} = \mathbf{y}'\mathbf{X}^{-1}\mathbf{p} = \mathbf{y}'\mathbf{q} = \mathrm{E}^Q(Y)$$

Therefore, equation (1.4) defines a *pricing function*, the price of Y is a function of Y. For convenience, we denote the pricing function by

$$\wp(Y) = \mathrm{E}^Q(Y) \tag{1.7}$$

Because the expectation operator is an increasing linear operator, the pricing function is an increasing linear function.

Recall that when we calculate the price of replicating portfolio Z in equation (1.1), we automatically take advantage of the linearity property of pricing function

$$\wp(aX + bY) = a\wp(X) + b\wp(Y)$$

for the absence of arbitrage requires that the pricing function must be linear.

D: State Price

When flipping a coin, the sample space is $\Omega = \{H, L\}$, where H and L signify simple events for heads and tails respectively. In financial economics, we say the world (market) has two possible *states*, state H (heads) and state L (tails), denoted as $w_1 = H$ and $w_2 = L$. Thus, we see that there is a one-to-one correspondence between simple event in probability and state in financial market.

The *state payoff* is one unit if a particular state occurs and zero in all the other states. For example, let I_2 be the state payoff of state L, then

$$I_2 = \begin{cases} 0 & w = H \\ 1 & w = L \end{cases}$$

Mathematically, state payoffs are denoted as

$$I_i \equiv \mathit{1}_{w=w_i} = \mathit{1}(w = w_i) \qquad i = 1, 2$$

where $\mathit{1}(\cdot)$ is an indicator function, $\mathit{1}(x) = 1$ if x is true, and otherwise $\mathit{1}(x) = 0$.

Let

$$g_i = \wp(I_i)$$

be the price of state payoff I_i, then the *state prices* are

$$g_1 = \wp(I_1) = \mathrm{E}^Q(I_1) = 1 \cdot q + 0 \cdot (1-q) = q$$
$$g_2 = \wp(I_2) = \mathrm{E}^Q(I_2) = 0 \cdot q + 1 \cdot (1-q) = 1-q$$

it is very interesting that the random variable I_i is corresponding to standard basis vector $\mathbf{i}_i$ (the ith column of identity matrix $\mathbf{I}$). Furthermore, discrete random variables on finite sample space $\Omega = \{w_1, w_2, \cdots, w_S\}$ correspond one-to-one to vectors in Euclidean space $\mathbb{R}^S$. As $S = 2$ in our example, denote

$$\mathbf{y} = \begin{bmatrix} y_H \\ y_L \end{bmatrix} \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then

$$\begin{aligned} \wp(\mathbf{y}) \equiv \wp(Y) &= \wp(y_1 I_1 + y_2 I_2) \\ &= y_1 \wp(I_1) + y_2 \wp(I_2) \qquad \text{by linearity} \\ &= y_1 g_1 + y_2 g_2 \end{aligned}$$

the pricing function turns out to be the dot product of state price vector and payoff vector

$$\wp(Y) = \wp(\mathbf{y}) = \mathbf{y}'\mathbf{g} = \mathbf{y} \cdot \mathbf{g} \tag{1.8}$$

Given the numerical settings in Example 1.1.1, we have

$$g_1 = q = \frac{1}{3}$$
$$g_2 = 1 - q = \frac{2}{3}$$

The price of Y is

$$\wp(Y) = \mathbf{y} \cdot \mathbf{g} = y_1 g_1 + y_2 g_2 = 20 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = 8$$

E: Stochastic Discount Factor

The risk-neutral pricing formula (1.3) is computing in Q world probabilities. However, for some applications (in particular in empirical researches), it is common to write this expected value directly under the objective probability measure in P world: Let's define random variable Ψ by

$$\Psi = \begin{cases} \psi_1 = g_1/\mathrm{P}(w_1) & w = w_1 \\ \psi_2 = g_2/\mathrm{P}(w_2) & w = w_2 \\ \quad \vdots & \quad \vdots \\ \psi_S = g_S/\mathrm{P}(w_S) & w = w_S \end{cases}$$

or equivalently

$$\Psi(w_i) \equiv \psi_i = g_i/\mathrm{P}(w_i) \qquad w_i \in \Omega = \{w_1, w_2, \cdots, w_S\}$$

then the price of any asset Y is given by (assumed finite states)

$$\wp(Y) = \mathrm{E}(\Psi Y) \tag{1.9}$$

where Ψ is called a *stochastic discount factor* (SDF). Which reflects the fact that the price of an asset can be calculated by "discounting" the future payoff by the stochastic factor Ψ and then taking the expectation.

Remark: When the payoff Y is deterministic and the continuously compounded interest rate r is constant, $Y_0 = e^{-rt}Y = \mathrm{E}(e^{-rt}Y)$. We are familiar with the e^{-rt} as the discount factor for payoff 1 at future time t, the SDF Ψ generalizes this concept to random world.

The pricing function can be represented in many specific forms: Eq (1.7) using risk-neutral probability, Eq (1.8) using state price, and Eq (1.9) using stochastic discount factor. Eq (1.7) is most seen in pricing of derivatives, Eq (1.8) is often adopted in financial economics when the number of state is finite, and Eq (1.9) is widely used in empirical study of asset pricing, in particularly, when a separable and additive utility is chosen, Ψ is a marginal rate of substitution.

F: Some Comments

When the number of state is finite, vectors and random variables have one-to-one correspondence. Thus, linear algebra and probability theory can be used interchangeably. A portfolio Z consists of 3 shares of X and 4 shares of Y

$$Z = 3X + 4Y$$

has an equivalent vector form

$$\mathbf{z} = 3\mathbf{x} + 4\mathbf{y}$$

When the number of state is infinite, mathematics tools for infinite-dimensional vector space such as Hilbert space is needed. Hilbert space generalizes the notion of Euclidean space, and it is the important topic of functional analysis, where the analyses of both finite and infinite space are unified. However, details of these are beyond the scope of this book.

1.1.2 Linear Pricing Function

We have understood that if the market is free of arbitrage, the pricing function must be linear. We will show that the price of a knock-out perpetual option is seen immediately if we follow the linear pricing function, without the clumsy assumption of GBM (geometric Brownian motion). Furthermore, the put-call parity, is merely a simple exercise of the linearity of pricing function. Finally, in a horse racing example, we demonstrate that, if there does not exist a linear pricing function, there will be an arbitrage opportunity.

A: Perpetual Option

Since the pricing function must be homogeneous: $\wp(aX) = a\,\wp(X)$, where a is a real number and X is the payoff of the asset. The following example is a bit complicated if we try to solve it using Black-Scholes equation. However, we can find out the answer within a few seconds if we take advantage the homogeneity of pricing function.

Example 1.1.2: A perpetual option has no expiration date. It pays \$4 if CSI hits \$120, and ceases to exist. The price of CSI is 60, assume that CSI pays no dividends and hits \$120 in a finite time, the volatility is 0.3, and the risk-free rate is 0. Find the price of this perpetual option.

Let t be the first time that CSI hits 120, we have $C_t = 4 = S_t/30$ almost surely. Thus the price of the perpetual option is

$$C_0 = \wp(C_t) = \wp(S_t/30) = \wp(S_t)/30 = S_0/30 = 2$$

If you have written one perpetual option, how to manage the risk? Since $C_t = S_t/30$, you may buy $1/30$ shares of CSI at a cost of \$2. When CSI hits \$120, you sell the $1/30$ shares of CSI and use the proceeds \$4 to pay the option holder.

Remark: Finance is an art of cash flow transformations. Time and amount are the two vital factors of cash flow: the time can be a fixed time or a random time, the amount also can be a fixed amount or a random amount.

1. A fixed payoff at random time, even when risk-free rate is 0, is discounted!
2. Pricing function for payoff at different time should be different. For example, let the price of a payoff X at time t be $\wp_{0,t}(X)$, then

   $$\wp_{0,2}(1) = e^{-2r} \neq e^{-3r} = \wp_{0,3}(1)$$

   if the constant risk-free rate is $r > 0$.

B: Put-call Parity

Consider a (K, T) European call option (with strike price K and time to maturity T) on stock S, a corresponding European put, and a forward with delivery price K. We have the put-call parity

$$P_0 + S_0 = C_0 + Ke^{-rT}$$

where C_0, P_0, and S_0 are the prices of call, put and the underlying stock respectively, and constant r is the constant risk-free rate ($\wp(1) = e^{-rT}$, continuously compounded).

We do not need the assumption of geometric Brownian motion (GBM) on price dynamics of the underlying stock. What we need is that the pricing function is linear: At time T, let S_T be the price of the stock, then the payoff of the call is

$$(S_T - K)^+ = \max(S_T - K, 0)$$

the payoff of the put and forward are $(K - S_T)^+$ and $S_T - K$ respectively, where $x^+ \equiv \max(x, 0)$. It is obvious that

$$x^+ = (-x)^+ + x \qquad \forall x \in \mathbb{R}$$

thus

$$(S_T - K)^+ = (K - S_T)^+ + (S_T - K)$$

We see that a European call is a portfolio of a corresponding European put and a forward. Following the linearity of pricing function

$$\begin{aligned} C_0 &= \wp((S_T - K)^+) \\ &= \wp((K - S_T)^+ + (S_T - K)) \\ &= \wp((K - S_T)^+) + \wp((S_T - K)) \\ &= P_0 + \wp(S_T) - \wp(K) && \text{by } P_0 = \wp((K - S_T)^+) \\ &= P_0 + S_0 - K\wp(1) && \text{by } S_0 = \wp(S_T) \\ &= P_0 + S_0 - Ke^{-rT} && \text{by } \wp(1) = e^{-rT} \end{aligned}$$

By a simple rearrangement we reach the put-call parity.

C: Horse Racing

Stating *odds against* is a convenient way to propose a bet. When a bookmaker offers betting odds of 6:1 against some event occurring, it means that she is prepared to pay out a prize of six times the stake, and return the stake as well, to anyone who places a bet, by making the stake, that the event will occur. If the event does not occur, then the bookmaker keeps the stake.

Example 1.1.3 (horse racing): $N = 3$ horses with odds

$$o_1 = 5 \quad o_2 = 4 \quad o_3 = 2$$

Is there a betting scheme that results in a sure win (assume no ties, or else run again)?

As a numerical demonstration, let x_i be the stake on horse i

$$x_1 = 5 \qquad x_2 = 6 \qquad x_3 = 10$$

when the first horse wins (outcome/state 1), the gain

$$G_1 = (1 + o_1)x_1 - \sum_{i=1}^{N} x_i = 9$$

similarly, $G_2 = G_3 = 9$.

Since $\sum \frac{1}{1+o_i} = \frac{7}{10} \neq 1$, there is a betting scheme

$$x_i = \frac{(1 + o_i)^{-1}}{1 - \sum_{n=1}^{N}(1 + o_n)^{-1}} \qquad i = 1, 2, \cdots, N$$

which will always yield a gain of exactly 1: For any outcome j, the gain is

$$G_j = (1 + o_j)x_j - \sum_{i=1}^{N} x_i = \frac{(1 + o_j)(1 + o_j)^{-1} - \sum_{i=1}^{N}(1 + o_i)^{-1}}{1 - \sum_{n=1}^{N}(1 + o_n)^{-1}} = 1$$

There exists a sure win, why? In the market setting of horse racing, there are 4 assets: lottery on horse X_i, $i = 1, 2, 3$, and a risk-free asset (with $B_0 = B = 1$). There are 3 states, with state i being the outcome that the ith horse wins. For simplicity, assume that the lottery of each horse is \$1. Let's apply

Eq (1.8) to all the assets, we arrive the following system of linear equations

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.10}$$

where g_i is the ith state price. For example, g_2 is the the price of payoff $[0; 1; 0]$: if you like to get \$1 when the 2nd horse wins, you should bet g_2 on horse 2. Unfortunately, the system of linear equations (1.10) is inconsistent, for the rank of the augmented matrix is greater than the rank of the coefficient matrix. Therefore, we are not able find a linear pricing function in form of state price. In fact, taking a closer look, the pricing for the market setting of horse racing can not be linear: Let $\mathbf{x}_i$ denote the payoff of lottery i, $i = 1, 2, 3$, and $\mathbf{1}$ be the payoff of risk-free asset, then $\wp(\mathbf{x}_i) = 1$ and $\wp(\mathbf{1}) = 1$. We see that

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{6}\mathbf{x}_1 + \frac{1}{5}\mathbf{x}_2 + \frac{1}{3}\mathbf{x}_3$$

However

$$1 = \wp(\mathbf{1}) > \frac{1}{6}\wp(\mathbf{x}_1) + \frac{1}{5}\wp(\mathbf{x}_2) + \frac{1}{3}\wp(\mathbf{x}_3) = \frac{7}{10}$$

the pricing function will never be a linear one. Besides, if we try the way of risk-neutral pricing, we will find that there does not exist risk-neutral probability measure (Exercise 1.16).

If there does not exists a linear pricing function, can we always find an arbitrage opportunity? The answer is unambiguous yes, which is discussed in the coming section §1.3.

§ 1.2 Foundations of Finance

A deterministic world is the starting point to understand finance. A perfect bank is the best to depict the characteristics of the world with certainty. However, the essence of finance is its uncertainty. To understand the risky nature of finance, we prepare some backgrounds and present the standard assumptions on financial market.

1.2.1 Terminology

A starting point to understand the finance literature is the notation of *perfect bank*: A perfect bank never defaults, it has no transactions fees, no service charges, no spreads and no taxes. Its interest rate applies equally to any size of principal of deposits or loans, from 1 cent (or fraction thereof) to \$1 centillion (10^{303}, or even more). We do not say that interest rates for all maturities are identical. For example, a 5-year certificate of deposit (CD) might offer a higher rate than a 1-year CD. However, the 2-year CD must offer the same rate as a 2-year loan as their maturities are identical. The interest rate provide by the perfect bank is called the *risk-free rate*, denoted as $R_{0,t} = R(0,t)$ for a maturity of t years (starts from time 0 to t). The risk-free rate represents the interest that an investor would expect from an absolutely risk-free investment over a given period of time.

As a function of maturity h, the curve $Y_t(h) = R(t, t+h)$ is call a *term structure* (yield curve) of interest rates at time t, where $R(t, t+h)$ is the risk-free rate that starts at time t and lasts for h years. The shape of the yield curve may be normal, flat or humped, or even inverted. The dynamic evolution of the yield curve $Y_t(h)$ over time t is extremely complicated. A "stylized fact" is that yield curves tend to move in parallel (i.e., the yield curve shifts up and down as interest rate levels rise and fall). However, even $Y_0(h)$ is flat, $Y_1(h)$ may not be flat, it may take any typical shapes.

A: Risk-free Rate

How to understand that a risk-free rate is random? It is risk-free in the sense that what will be received at the end of the investment is known at the very beginning. Its payoff will not fluctuate because it is settled at the beginning of investment period. By holding to maturity, the investment has no uncertainty. Risk-free rate is random means that the future spot rate $R(t,T)$ is random, say, at time $s < t$, $R(t,T)$ is a random variable for any $T > t$.

The spot rate $R(t,T)$ is observed at time t, and apply for the period of time from t to T. The (simple) rate of return of risky asset $Y(t,T) = V_T/V_t - 1$ (when the value $V_t > 0$) is known at time T. At time $s < t$, the values of $R(t,T)$ and $Y(t,T)$ are unknown. At time t, $R(t,T)$ is known, yet $Y(t,T)$ is still unknown, as a random variable, $Y(t,T)$ is realized at time T. For risky assets, like stocks, the rates of return are known at the end of investment horizon. $R(t,T)$ is deterministic for holding from t to T, if the holding is ended before maturity date T, its rate of return is random.

Even when $R(t, T)$ is random, it is still called risk free. We take no risk (no effort, no loss) in earning R, known at the beginning time t, but R itself is varying over time variable t.

The interest rate is often denoted by r if interest rates for all maturities are identical, or the maturity is clear in the context. And when we say that the interest rate is a constant r, we assume that $Y_t(h) = r$ for any time t and any maturity h. Unless explicitly stated, otherwise for simplicity we assume that the interest rate is a constant r in the following text.

Any asset that is growing at risk-free rate is a *risk-free asset*. For examples, an account of perfect bank with constant interest rate, and a zero-coupon treasury bond are often taken as risk-free asset. Because we can undoubtedly put the money in the pocket, at an uninteresting risk-free rate $r = 0$, we always assume that there is a risk-free bond with price $B_0 = 1$ unless explicitly ruled out.

B: Payoff

The *payoff* of an asset is the value of the asset at some future time, or the future amount paid to acquire the asset. Which equals the quoted price plus embedding rights (dividend, coupon, etc.) For examples, for a stock with price S_0, its payoff at $t = 2$ is S_2 if there is no dividend[2] from time 0 to 2. The payoff of a bond right at time 1 (first coupon date) is the price of the bond at time 1 plus the coupon at time 1, thus the payoff at time 1 is greater than the cash flow at time 1 (coupon) of the bond when it does not mature. Immediately after the payout of coupon, the payoff of the bond is the market price of the bond. An asset is characterized by its payoff, thus, we use random variable X, to refer to both asset X and payoff X. Please note that different assets who have the same payoff are indifferent in the context of asset pricing. Seeing this way, a portfolio is a composite asset; And an individual asset is a simple portfolio.

The payoff of a portfolio is the linear combination of its constituent assets' payoff. For example, a portfolio Z consists of a shares of X and b shares of Y has a payoff of

$$Z = aX + bY$$

at the cost of

$$Z_0 = aX_0 + bY_0$$

thus, the price of a portfolio observes some form of linear pricing function:

$$\wp(aX + bY) = \wp(Z) = Z_0 = aX_0 + bY_0 = a\wp(X) + b\wp(Y)$$

C: Price

The notation of *price* in finance has many interpretations:

1. The price of a security (capital asset), like the IBM stock price, which is the trading price.

[2] If there is some dividend between time 0 and 2, the payoff at time 2 dependends on how the dividend is invested. Unless otherwise stated, we assume that all income from the asset is reinvested in itself when there is any reinvestment.

2. The cost of capital: Thinking of an interest rate as the cost of money, as a price paid, then the expected or required rate of return of a stock is defined to be the cost of equity. However, I do not think that such definition is proper, because the shareholder and creditor have different rights. The expected or required rate of return is not binding, not an obligation like the cost of debt.

In this book, we refer the price of an asset as the trading price of the asset. The price P_t at some future time t may less than the payoff at time t if there is a cash dividend, since its value is the sum of price P_t and the cash dividend.

1.2.2 Analytical Theories of Finance

The analytical theories of finance are

1. The time value of money: The idea is simple—a dollar today is worth more than a dollar tomorrow. Due to the time preference, we have $0 < \wp(1) \leqslant 1$, and the (simple) risk-free rate of return $r = 1/\wp(1) - 1 \geqslant 0$ is nonnegative.
2. Diversification: This central idea is straightforward. If investors want to make as much money as they can with the least risk, they should diversify. Why? “Don’t put all your eggs in one basket”. Because diversification reduces risk (variation). The intuition here is that with a large portfolio holding, some assets will do very well, some will do very badly and most will perform not far away from expectations. Those that do very well will mostly cancel out those that do very badly, the fluctuation for the portfolio as a whole will be smooth and show less variation. Thus, given the expected return, a collection of investment assets selected carefully has collectively lower risk than any individual asset.
3. Arbitrage-free pricing: The simple idea here is that “no making something out of nothing” or “no free lunch”. If there are two ways to get the same payoff they must have the same price, or else an arbitrage opportunity would exist. Thus, a security must have a single price, no matter how that security is created. Furthermore, the pricing function must be linear, and for any weak-positive payoff (no loss and a chance of gain), the price should be positive.
4. Behavioral finance (emerging): Behavioral finance examines what the realities of market participants are, and explains the financial phenomena from the characteristics shown by market participants. Behavioral finance is a relatively new field, it studies the *sub-rational* financial decisions by considering the effects of psychological, social, cognitive, and emotional factors, and the consequences for market prices, returns, and resource allocation.
5. Corporate finance (developing): The primary goal of corporate finance is to maximize or increase the market value (shareholder value). All other goals of the firm are intermediate ones leading to firm value maximization, or operate as constraints on firm value maximization. The main concepts in the study of corporate finance are associated with: how to pick projects; how to finance them; how much to pay in dividends, and how to distribute the rights and responsibilities among different

participants in the corporation.

Historically, valuation is rooted in the idea of the time value of money. The idea is centred on the work of Fisher (1930) in *The Theory of Interest*. Diversification is mentioned in the Bible, the modern understanding of diversification dates back to the Nobel Prize winning work of Markowitz (1952). Finally, arbitrage roots in the instinct of investors, it remains hidden for a very long time, and only makes its presence the first time due to the pioneering work of Modigliani and Miller (1958).

There are more subjective and mental activities involved in the field of behavioral finance and corporate finance, thus, analytical models are easily lost in their complexity on these challenging issues. A general framework capable of providing a quantitative assessment is still lacking.

A: Time Value

We assume that there is a risk-free asset, CD (bank account, or money-market account) or risk-free bond grows from $B_0 = 1$ to B_t at time t. Let $D_{0,t}$ be the price at time 0 for a discount bond (zero-coupon bond) with payoff 1 at time t, then $D_{t,t} = 1$. Since interest rates are nonnegative (deterministic or random), we have $B_u \geqslant B_t$ and $D_{0,u} \leqslant D_{0,t}$ if time $u > t$.

Suppose that the world of interest rate is deterministic:

- If the interest rate is a constant $r > 0$ (flat), then

$$B_t = e^{rt} = 1/D_{0,t}$$

- If the future risk-free rate is time varying: The price of the risk-free bond at time t is (continuous compounding)

$$B_t = e^{tR_{0,t}} = 1/D_{0,t}$$

However, the real world is not a world with certainty, and the future risk-free rates are uncertain. When the future risk-free rates are characterized by random variables, please note that (Eq 4.17)

$$B_t \neq 1/D_{0,t}$$

for at time 0, B_t is a random variable, and (Eq 4.14)

$$D_{0,t} = \wp_{0,t}(1) = e^{-tR_{0,t}}$$

is a known number.

Remark: For risk-free asset B_t and $D_{0,t}$, B_t is a theoretical bond while $D_{0,t}$ is zero-coupon bond traded (maturity usually less than one year). B_t grows like a current account, the interest rate for next year only knows at the beginning of next year. $D_{0,t}$ works like a fixed account deposit, and grows to 1 at time t. B_t records history from time 0 to t, while $D_{0,t}$ looks into future time t at time 0.

B: Modern Portfolio Theory

The modern portfolio theory is based on the mean-variance preference, a simplify criterion using only the first two moments of the probability distributions of risky returns. Supposed that the mean vector and variance matrix of a group of primitive risky assets are known, by seeking the lowest variance for a given expected return or seeking the highest expected return for a given variance level, an efficient frontier emerges. The key insight of modern portfolio theory is that an asset's risk and return should not be assessed by itself, but by how it contributes to a portfolio's overall risk and return. Because of the diversification frontier, the trade-off between risk and return will be the same for all investors. The two-fund separation theorem (Tobin, 1958) is a consequence of diversification and mean-variance preference.

1.2.3 Perfect Market

We assume the following institutional facts:

1. Perfect market trading: frictionless transactions, and the trading price is the equilibrium price
   - There are no bid-ask spreads, i.e., the selling price is equal to the buying price for any asset.
   - There are no transaction costs of trading.
   - There are no taxes.
   - There are no bankruptcy costs.
2. Perfect market liquidity: buy and/or sell any quantity instantaneously at market price.
   - Short positions[3] ($h = -2$ means sell short 2 shares), as well as fractional holdings, are allowed. In mathematical terms this means that every $\mathbf{h} \in \mathbb{R}^N$ is an allowed portfolio of N assets.
   - The market is completely liquid, i.e. it is always possible to buy and/or sell unlimited (not infinite) quantities on the market. In particular it is possible to borrow unlimited amounts from the bank (by selling bonds short).
3. Perfect market information: asset prices fully reflect all available information.
   - The information set of future states and probabilities are known.
   - Investors are rational and fully grasp the current and past information.
   - There are no costs to obtain and process information immediately.

The perfect capital market assumptions are clean test tubes, an antiseptic world. By starting from this immaculate laboratory equipment, financial researchers can analyze the consequences of their models, test their hypotheses, and determine how closely their theory accords with the real world.

The further expansion of financial theory beyond the known territories is the process of relaxing the perfect market assumptions. For instance, the behavioral finance is primarily concerned with the rationality of investors. Emotion and psychology influence our decisions, we make decisions based on approximate rules of thumb and not strict logic, thus we behave in unpredictable or even somewhat irrational ways, such as herd behavior, overconfidence and overreaction.

[3] A short position of bond can be regarded as borrowing money. And a short position in a stock is done by borrowing the stock and selling it on the market and buying back in some future time. Short positions are mostly settled in cash.

In my opinion, the efficient markets hypothesis[4] (EMH) is not a core idea of finance. EMH is only part of perfect market assumption with agents who are fully rational and informed, such that all past information available up to time t is impounded in the current price.

1.2.4 Postulates on Asset Prices

Asset price is the key variable in financial market. First and foremost, the marketable primitive security must have a single price as trading price. Where we define a *primitive security* to be an instrument such as a stock or bond for which payments (dividends, coupons, or yield) depend only on the financial status of the issuer.

Postulate 1 (**Law of One Price**)

Each primitive security has a unique price.

The primitive securities can be priced. Let X be the set of payoffs of primitive securities, the correspondence between X and security prices is a function denoted by $\wp : \mathsf{X} \to \mathbb{R}$. Please note the pricing function $\wp$ is applied to all securities, not each primitive security has its own pricing function.

Portfolio is an essential concept in finance, where a collection of securities are held and treated as a whole. The payoff and price of a portfolio is formed by linear combination.

Postulate 2 (**Law of Linear Combination**)

The payoff of a portfolio equals the linear combination of payoffs of its constituent assets. The price of a portfolio equals the linear combination of prices of its constituent assets.

The law of linear combination asserts that the pricing function is linear: For portfolio Z created by a shares of asset X and b shares of asset Y, we have the payoff $Z = aX + bY$ and thus

$$\wp(aX + bY) = \wp(Z) = a\wp(X) + b\wp(Y)$$

Immediately, we find that the pricing functions always go through the origin. If the payoff of an asset is zero, the price must be zero.

Proposition 1.1: $\wp(0) = 0$.

Since a portfolio is a linear package, the payoff space is a vector space, for brevity, we still use symbol X to denote the linear combination of payoffs of primitive assets. At the same time, the pricing function $\wp$ is extended to the whole payoff space X. The law of linear combination assures that the pricing function is linear and thus zero-axial.

Remark: Portfolios do not add value today, since the whole equals the sum of its parts. However, portfolios change the future, reduce the volatility, the future values become less unexpected downs and

[4] The phrase of efficient markets hypothesis is descriptive, it is designed more to capture an intuition than to state a formal mathematical result. The definition of information efficiency is purposely vague, the exact mechanism by which prices incorporate information is still a mystery.

reduce expected ups. We need portfolios for a better secure future without worsening today.

The financial markets place certain restrictions on pricing functions. We will begin by defining some mathematical symbols: A random variable X is said to be *weak-positive*, if $X \geqslant 0$ (more exactly, $\mathrm{P}(X \geqslant 0) = 1$) and $\mathrm{P}(X = 0) < 1$. Thus, if X is weak-positive (in the sense of almost surely), X is non-negative, and can not constantly be zero. We write $X \gneqq 0$ when X is weak-positive, where the symbol "$\gneqq$" means *weak greater than*[5]. An asset with weak-positive payoff is called a limited liability, because the holder's loss is the initial investment at most.

Postulate 3 (**Limited Liability**, or **Positivity**)

For any limited liability portfolio, the price is positive.

The postulate of positivity is a summary of market practices that define the distinctive property of a pricing function, which have more sense of financial contract than of mathematics. Postulate 3 reveals many market conventions:

1. In real world, all of the primitive securities are limited liabilities, their payoffs are weak-positive. Thus, for any primitive risky asset, the price is positive.
2. For any positive constant payoff, the price is positive. Thus, the price of risk-free assets are positive, in particular $\wp(1) > 0$, which ensures that the risk-free interest rates are well defined.

A claim right entails obligations, thus if $X \gneqq 0$ then $\wp(X) > 0$. Otherwise, if $X \gneqq 0$ and $\wp(X) = 0$, we can get something for nothing, there exists an arbitrage opportunity.

1.2.5 Self-Financing Portfolio

Trading is the magic of financial market. Intermediate tradings create new possibilities beyond buy-and-hold portfolios, thus enlarge the payoff space of market.

A: Pricing Function for Static Portfolios

Static portfolio is the portfolio that we buy and hold. The market portfolio is an example of static portfolio, a buy-and-hold portfolio, where the numbers of holdings remain constant[6] since the initial forming of the portfolio. Static portfolios are characterized by no intermediate tradings. The value of a static portfolio grows automatically, even the numbers of holdings remain unchanged, the value percentage (value weighted, if the portfolio value is always positive) of each constituent asset may change due to the variations of asset prices.

[5] In theory of probability, it means *statewise dominance*, also known as *state-by-state dominance*. $X \gneqq Y$ is read as X is statewise dominance over Y. In this sense, for real numbers $x, y \in \mathbb{R}$: $a \gneqq b \iff a > b$; and for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$: $\mathbf{x} \gneqq \mathbf{y} \iff \mathbf{x} \geqslant \mathbf{y}$, and $\mathbf{x} \neq \mathbf{y}$.

[6] The number of shares on a stock paying stock dividends will be changed in a buy-and-hold strategy.

In continuous time or T-step discrete time model[7], each primitive security has a price dynamics modelled by a stochastic process where intermediate tradings are allowed. Given any time points t and u in $[0, T]$, if P_t is the price process of any primitive asset (one unit at time 0), there exists a pricing function $\wp_{t,u}(\cdot)$, such that

$$P_t = \wp_{t,u}(P_u) \qquad 0 \leqslant t \leqslant u \leqslant T$$

Note that $\wp_{t,t}(\cdot)$ is an identity function. Let $D_{t,T}$ be the price at time t for a discount bond with payoff 1 at time T, then $D_{t,T} = \wp_{t,T}(1)$ and $D_{T,T} = 1$. It is straightforward that if V_t is the value process of a static portfolio, we have

$$V_t = \wp_{t,u}(V_u) \qquad 0 \leqslant t \leqslant u \leqslant T$$

Remark: With respect to forms of function, all pricing function in the collection of $\{\wp_{t,u}(\cdot) : 0 \leqslant t \leqslant u \leqslant T\}$ are somehow connected. For example, for any s, t and u with $0 \leqslant s \leqslant t \leqslant u \leqslant T$

$$\wp_{s,u}(\cdot) = \wp_{s,t}(\wp_{t,u}(\cdot))$$

$\wp_{s,u}(\cdot)$ is a composite function of $\wp_{s,t}(\cdot)$ and $\wp_{t,u}(\cdot)$ for any t in $[s, u]$. The representation of $\wp_{t,u}(\cdot)$ will be presented in Formula (3.21) where additional knowledge on conditional expectation are needed.

Given payoff X at time T, let X_t be its price at any time $t \leqslant T$, then

$$X_t = \wp_{t,u}(X_u) = \wp_{t,u}(\wp_{u,T}(X)) = \wp_{t,T}(X) \qquad 0 \leqslant t \leqslant u \leqslant T$$

Which states that at time u, $X_u = \wp_{u,T}(X)$ is the price of future payoff X; At time $t < u$, X_u is a payoff (random variable). For the sake of clarity, we drop one or all subscripts such as $\wp_t(X_u) = \wp_{t,u}(X_u)$, or even the shortest form

$$\wp(X_t) = \wp_0(X_t) = \wp_{0,t}(X_t)$$

whereas it is clear from the context.

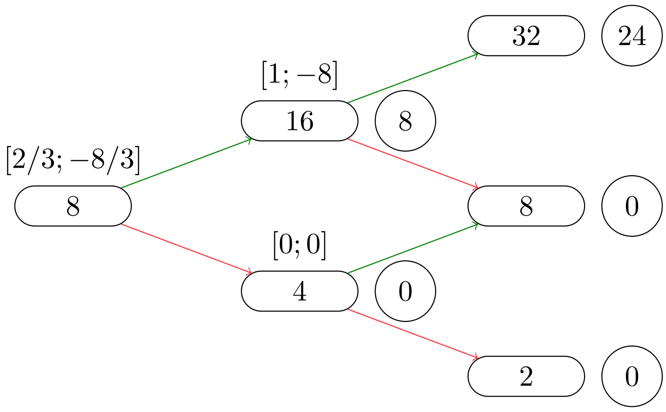
The following Example 1.2.1 shows that some payoffs can not be reached by a static portfolio, but can be reached by dynamic portfolios.

Example 1.2.1: We have a stock presently priced at \$8. In exactly one month the stock price will be either \$16 or \$4 with equal possibility. At the end of second month, the stock price will be either double or half again. The current interest rate is 0 compounded monthly. Let X be the payoff of a European call option on the stock with a strike price \$8 expiring in two months, then there is no static portfolio to replicate the European call.

The payoff X and the dynamics of stock price are depicted in Figure 1.1: Suppose that we are able

[7] For continuous time model, trades can be done continuously. For T-step discrete time model, trades are allowed only in time points $t \in \{0, 1, 2, \cdots, T\}$.

Figure 1.1: Static and Dynamic Portfolios The stock price will be either double or half each month, there is no static portfolio to replicate a European call expiring in two months. However, a dynamic replicating portfolio exists.



to buy a shares of stock and b shares of risk-free bond to replicate X, then

$$32a + b = 24$$

$$8a + b = 0$$

$$2a + b = 0$$

which has no solution. However, there is a dynamic replicating portfolio: at $t = 0$, we long $2/3$ shares of stock and short $8/3$ shares of risk-free bond, denoted by $[2/3; -8/3]$. At $t = 1$, if the stock price is double to be \$16, we rebalance the portfolio to $[1; -8]$

$$\frac{2}{3} \cdot 16 + \left(-\frac{8}{3}\right) \cdot 1 = 8 = 1 \cdot 16 + (-8) \cdot 1$$

we buy additional $1 - \frac{2}{3} = \frac{1}{3}$ shares of stock at a cost of $\$\frac{16}{3}$, which is funded by short sell an additional amount of $8 - \frac{8}{3} = \frac{16}{3}$ shares of risk-free bond. The payoff of portfolio $[1; -8]$ holding to $t = 2$ is either 24 or 0. Similarly, if the stock pice is cut half to be \$4, the value of the portfolio $[2/3; -8/3]$ formed at $t = 0$ holding to $t = 1$ is 0. We liquidate the stock to cover the short positions of bonds at $t = 1$, our new portfolio is $[0; 0]$, whose payoff is 0 at $t = 2$.

Static portfolios are essential, however they are limited due to the exclusion of intermediate tradings. Intermediate tradings will expand the dimension of marketable payoff space. Dynamic portfolios which primarily interest us are the self-financing portfolios, where the addition of any asset has to be financed through the sale of some other asset.

B: Self-Financing Dynamic Portfolio

When there are intermediate tradings, we employ a continuous time or T-step discrete time model. In a T-step discrete time model, portfolios can be rebalanced at the intermediate steps $t = 1, 2, \cdots, T-1$.

Assume that the market has $N+1$ primary assets, the last one is a risk-free bond and the others are risky assets. Let $\mathbf{h}_t = [h_{t1}; \cdots ; h_{tN}; h_{t0}]$ be the portfolio at time t, where h_{t0} is the number of risk-free bond, and h_{ti} is the number of shares of risky asset i. We hold portfolio $\mathbf{h}_t$ at period $t+1$ for the time interval $(t, t+1]$. The portfolio $\mathbf{h}_t$ was created at time t, and held until time $t+1$. We allow the holdings $\mathbf{h}_t$ to be a *predictable* contingent strategy, i.e. the portfolio we buy at time t is allowed to depend on all information up to time t, by observing the evolution of the assets' prices. We are, however, not allowed to look into the future.

Let P_{ti} be the price of security i at time t, $\mathbf{p}_t = [P_{t1}; \cdots ; P_{tN}; B_t]$ be the price vector, and $V_t = V(\mathbf{h}_t) = \mathbf{h}'_t\mathbf{p}_t$ be the value of portfolio $\mathbf{h}_t$. Then, a portfolio rebalancing from $\mathbf{h}_{t-1}$ to $\mathbf{h}_t$ is *self-financing* at time t if

$$\sum_{i=0}^{N} h_{t-1,i}P_{ti} = \mathbf{h}'_{t-1}\mathbf{p}_t = \mathbf{h}'_t\mathbf{p}_t = \sum_{i=0}^{N} h_{ti}P_{ti} = V_t \qquad t = 1, 2, \cdots, T-1 \tag{1.11}$$

where the holdings h_{ti} and asset prices P_{ti} are all bounded, and $\mathbf{h}_t$ is predictable (known at time t).

The above condition (1.11), $\mathbf{h}'_{t-1}\mathbf{p}_t = \mathbf{h}'_t\mathbf{p}_t$, is simply a budget equation. It says that, at each time t, immediately after the assets' new prices are observed, we sell the entire "old" portfolio $\mathbf{h}_{t-1}$ (which was created at time $t-1$, and held until time t), and use the revenues to acquire the new portfolio $\mathbf{h}_t$. In other words, the cost of constructing the new portfolio is exactly offset by the proceeds of selling the old portfolio. For a better understanding of self-financing condition (1.11), we imagine the following steps

1. At time t, given the new price, the value of old portfolio is
$$V_t = \sum_{i=0}^{N} h_{t-1,i}P_{ti} = \mathbf{h}'_{t-1}\mathbf{p}_t$$
2. Let $t+0$ denote the moment right after the portfolio rebalancing at time t. Since the portfolio holding of assets changes from $\mathbf{h}_{t-1}$ to $\mathbf{h}_t$, the new portfolio value at time $t+0$ becomes
$$V_{t+0} = \sum_{i=0}^{N} h_{ti}P_{ti} = \mathbf{h}'_t\mathbf{p}_t$$
3. Suppose we adopt the self-financing policy, we must have
$$V_{t+0} = V_t$$

If the portfolio rebalancing from $\mathbf{h}_{t-1}$ to $\mathbf{h}_t$ observes the self-financing condition (1.11), there is no addition or substraction of fund from the portfolio rebalancing. Thus, in a self-financing rebalance, the purchase of additional units of one particular security cannot but be financed by the sales of other securities within the portfolio.

Definition 1.2: In a T-step discrete time model, a portfolio is *self-financing* if each rebalancing is self-financing. Let $\mathbf{h}_T = \mathbf{h}_{T-1}$, the holding sequence $\{\mathbf{h}_t\}_{t=0}^{T}$ of a self-financing portfolio is called a *self-financing trading strategy* or *portfolio strategy*. Given the price process $\{\mathbf{p}_t\}$, $V_t = \mathbf{h}'_t\mathbf{p}_t$ is the *value process* of the self-financing portfolio $\{\mathbf{h}_t\}$.

Remark: Static portfolios are self-financing portfolios. We do not need to discuss self-finance issue in one-step model, for there is no chance of portfolio rebalancing.

A portfolio is self-financing if there is no inflow or outflow of money after the initial forming of the portfolio. When rebalancing the portfolio, the purchase of a new asset must be financed by the sale of an old one in the portfolio. A self-financing portfolio is characterized by that the values of the portfolio just before and after any transactions are equal.

Remark: A self-financing portfolio observes $V_{t+0} = V_t$ at any time t

- Let $\Delta \mathbf{h}_t = \mathbf{h}_{t+1} - \mathbf{h}_t$ (forward difference, for backward difference is deterministic at time t) then

$$\mathbf{p}_t' \Delta \mathbf{h}_{t-1} = 0 \qquad \forall t$$

  Which states that increasing the number of any asset is done by selling other assets.

- Transactions do not change the value of portfolio

$$V_{t+0} - V_t = \mathbf{p}_t' \Delta \mathbf{h}_{t-1} = 0$$

  The rebalancing of portfolio do not change the value of a portfolio

- Gain of value process: For

$$\Delta V_t = V_{t+1} - V_t = \mathbf{h}_t' \mathbf{p}_{t+1} - \mathbf{h}_t' \mathbf{p}_t = \mathbf{h}_t' \Delta \mathbf{p}_t \tag{1.12}$$

  the change of a portfolio's value is caused by the change of its constituent assets' price, not by the change of the holdings

Let $V_t = \mathbf{h}_t' \mathbf{p}_t$ be the value process of a self-financing portfolio $\{\mathbf{h}_t\}$, then

$$\wp_{t,t+1}(V_{t+1}) = \wp_{t,t+1}(\mathbf{h}_{t+1}' \mathbf{p}_{t+1}) = \wp_{t,t+1}(\mathbf{h}_t' \mathbf{p}_{t+1}) = \mathbf{h}_t' \wp_{t,t+1}(\mathbf{p}_{t+1}) = \mathbf{h}_t' \mathbf{p}_t = V_t$$

Furthermore, given any time points t and u with $0 \leqslant t < u \leqslant T$, we have

$$V_t = \wp_{t,u}(V_u) = \wp_{t,u}(\wp_{u,T}(V_T)) = \wp_{t,T}(V_T) \qquad 0 \leqslant t \leqslant u \leqslant T$$

Thus, if payoff X at some future time T is given, and there is a self-financing portfolio $\mathbf{h}_t$ such that $X = V_T$, then for any $t \leqslant T$, $V_t = \wp_{t,T}(V_T)$ is equal to the price process of future payoff X.

When we model financial markets in continuous time, we employ stochastic calculus. In addition with other regularities, a self-financing portfolio is defined to be

$$\mathrm{d}V_t = \mathbf{h}_t' \mathrm{d}\mathbf{p}_t$$

the limit of condition (1.12) as $\Delta t \to 0$. Which will be further investigated as condition (7.7) and (7.8) in §7.3 for the Black-Scholes model.

§ 1.3 Arbitrage

It is difficult to understand exactly how to measure and price financial risk[8] for securities. Arbitrage-free pricing bypasses this problem successfully, which is a theory of pricing of securities that does not rely on attitudes towards risk. One of the major advances in financial economics has been to clarify and formalize the exact meaning of arbitrage. No-arbitrage principle uncovers a hidden relationships in asset prices: The absence of arbitrage is equivalent to the existence of a *positive* linear pricing function.

1.3.1 Arbitrage Opportunity

Let the value of a self-financing portfolio $\{\mathbf{h}_t\}$ at time t be $V_t = V(\mathbf{h}_t)$. If V_t is weak greater than 0, there is $V_t \gneqq 0$, or equivalently, $V_t \geqslant 0$ and $\mathrm{P}(V_t > 0) > 0$, V_t is non-negative and have a chance to be positive. If V_t is weak-positive and the expectation exist, there is

$$V_t \geqslant 0 \text{ and } \mathrm{E}(V_t) > 0$$

which means there is no chance of loss, and there is a sure expectation of gain.

A self-financing value process $\{V_t\}$ is denoted as portfolio (V_0, V_t) when we are most interested in the initial price and its value at time t.

A: Definition

An arbitrage opportunity makes something out of nothing: there are three ways

1. Getting money without future obligations.
2. Making money with no initial investment without any possibility of future loss and a possibility of future gain.
3. A mixed of the above two ways.

More exactly, given some future time point $t > 0$ and a self-financing value process V_t, we say that the portfolio (V_0, V_t) is an *arbitrage opportunity* , or an *arbitrage portfolio*[9], if

$$\begin{bmatrix} -V_0 \\ V_t \end{bmatrix} \gneqq 0 \tag{1.13}$$

Clearly, condition (1.13) contains three cases (disjoint subsets)

1. The *immediate arbitrage opportunity* turns something into nothing

$$V_0 < 0, V_t = 0 \tag{1.14}$$

The investors can get paid immediately and go away without any obligations.

[8]Volatility/variance is not always a risk, it can be a sweetener. Risk is often understood to be the potential for financial loss and uncertainty about its extent.

[9]An arbitrage portfolio may not be a zero-investment portfolio. Note that some textbooks define zero-investment portfolio as arbitrage portfolio.

2. The *zero-investment arbitrage opportunity* turns nothing into something

$$V_0 = 0, V_t \gneq 0 \tag{1.15}$$

Which states that the value of the portfolio is 0 at time 0, and we know[10] at time 0 that $V_t \gneq 0$ for sure. If $V_t > 0$, the future payoff is strictly positive. It is called a *strong arbitrage opportunity*. Furthermore, if $V_t = a > 0$ for some constant a, it is called a *naïve arbitrage opportunity*.

3. The *mixed arbitrage opportunity* is a double happiness

$$V_0 < 0, V_t \gneq 0$$

It is a portfolio of immediate and zero-investment arbitrage opportunities.

The immediate arbitrage opportunities and naïve arbitrage opportunities are called *deterministic arbitrage opportunities*, or *risk-free arbitrage opportunities*. Some textbooks and articles on finance define arbitrage opportunity only to be deterministic arbitrage opportunity, the more interesting cases in condition (1.15) are excluded.

A risk-free asset shifts cash flows across time. When there exist risk-free assets, an immediate arbitrage opportunity is easily converted to a zero-investment arbitrage opportunity. Thus, arbitrage opportunity can be defined by condition (1.15) if the risk-free asset exists (or if there is an asset that always has positive value).

Remark: The condition (1.13) for an arbitrage opportunity is defined in the sense of a long position (long the portfolio at time 0 and close it at future time t). Thus, if $V_0 > 0$ and $V_t = 0$, portfolio (V_0, V_t) will not be an arbitrage opportunity by definition. However, its short position, portfolio $(-V_0, -V_t)$ is an arbitrage opportunity.

Example 1.3.1: For a self-financing value process V_t, with $V_0 > 0$ and $V_t = 0$ for some $t > 0$, is there an arbitrage opportunity in the market?

Let portfolio X be the short position of asset V_t, then

$$X_0 = -V_0 < 0$$

and $X_t = -V_t = 0$, which is an arbitrage opportunity by definition.

B: Properties

An arbitrage opportunity results in a weak-positive value adding (no chance of depreciation)

$$V_t - V_0 \gneq 0 \tag{1.16}$$

However, condition (1.16) does not mean that we make something out of nothing, thus it does not assure the existence of an arbitrage opportunity.

[10] As a random variable, V_t is fully known at time t. However, at time 0, it is not completely unknown, the whole set of values is known. Therefore, some properties maybe unambiguous, such as weak positiveness.

A self-financing value process V_t is called an *arbitrage process* if it contains an arbitrage opportunity. It is a simple fact that when portfolio (V_0, V_t) is an arbitrage opportunity, portfolio (hV_0, hV_t) is an arbitrage opportunity only when $h > 0$. Thus, given an arbitrage process V_t, hV_t is an arbitrage process if $h > 0$. Furthermore, given a finite collection of arbitrage opportunities $\{(V_{0i}, V_{ti})\}$, any *convex combination* ($h_i \geqslant 0$, and $\sum h_i = 1$) forms an arbitrage opportunity

$$V_0 = \sum_i h_i V_{0i} \qquad V_t = \sum_i h_i V_{ti}$$

due to $[-V_0; V_t] \gneqq 0$. We have known that an arbitrage opportunity is positive scalable, and convex combinational. Thus, a *weak-positive combination* ($h_i \geqslant 0$ and $\sum h_i > 0$, or $\mathbf{h} \gneqq 0$) of arbitrage opportunities is an arbitrage opportunity.

Since there is a risk-free asset, if there exist an arbitrage opportunity, such that $V_0 = 0$, and $V_t \gneqq 0$. Then we are able to construct a portfolio X with $X_0 = 0$, and $X_T \gneqq 0$ for any time $T > t$. Thus, on detecting an arbitrage opportunity, we are adequate and easy to be interested only in the portfolio values at the two end points time 0 and T.

The following statements are equivalent

1. The market is *absence of arbitrage*.
2. There does not exist self-financing value process satisfy condition (1.13).
3. For any self-financing value process with $V_0 \leqslant 0$ and $V_t \geqslant 0$, we must have $V_t = V_0 = 0$.

When the market is absence of arbitrage, there does not exist any arbitrage portfolio, there does not exist any arbitrage process: For any self-financing value process, portfolio (V_0, V_t) or $(-V_0, -V_t)$ can not be an arbitrage opportunity.

1.3.2 No-Arbitrage Principle

Many important results of financial markets are based squarely on the hypothesis of no arbitrage.

No-Arbitrage Principle

There do not exist arbitrage opportunities in financial market.

An arbitrage opportunity represents a money pump, absence of arbitrage is more primitive than equilibrium. Investors prefer more money to less (insatiability axiom), the more the better. In perfect market, with frictionless transaction and complete liquidity and market information, suppose there is an arbitrage opportunity, no matter how small, a number of investors will be active, they will pursue arbitrage profits effectively and make the market free of arbitrage opportunities. Although financial markets are not so perfect, but compare to goods market or labor market, friction and illiquidity are least in financial markets. From another point of view, the rights and obligations are interdependent and inseparable, there is no right without obligation, nor is obligation without right. For the above reasons, arbitrage opportunities rarely exist in practice, no-arbitrage is applied to financial markets as a principle.

The postulates on asset prices (§1.2.4) wipe out all kinds of arbitrage opportunities

1. The law of one price and law of linear combination result in $\wp(0) = 0$, which rules out immediate arbitrage opportunities.
2. The positivity postulate requires that if $X \gneq 0$ then $\wp(X) > 0$, which rules out zero-investment arbitrage opportunities and mixed arbitrage opportunities.

Clearly, under this set of postulates, the market is free of arbitrage. On the other hand, if we assume the no-arbitrage principle, then the positivity postulate is true. Thus, given the law of one price and law of linear combination, the no-arbitrage principle is equivalent to the positivity postulate.

The most important implication of the absence of arbitrage is the existence of a positive linear pricing function. This finding, known as the first **fundamental theorem of asset pricing** (FTAP), was a milestone in financial economics.

Theorem 1.3 (First Fundamental Theorem of Asset Pricing): *In a perfect market, the market is free of arbitrage if and only if the linear pricing function is a positive function.*

Proof: see the appendix.

When the market is free of arbitrage, there is a positive pricing function, but may not be unique. When the pricing function is not unique, these pricing functions will give the same price to each existing asset, but they may set different prices to a newly introduced asset. The absolute price of the new asset is determined by the market and picked from the interval of arbitrage-free prices of its payoff.

Remark: We limit the number of assets in the market to be finite. In a finite market, it is always continuous for linear pricing function. In an infinite market, a linear function may fail to be continuous, then continuity is needed for FTAP in the linear pricing function.

A: Absence of Arbitrage

The postulates on asset prices define the following properties of pricing function. For any $X, Y \in \mathsf{X}$

1. Existence (assets can be priced): $\wp : \mathsf{X} \to \mathbb{R}$
2. Linearity (package vs unpack): $\wp(aX + bY) = a\,\wp(X) + b\,\wp(Y)$
3. Positivity (increasing): $\wp(X) > 0$ for $X \gneq 0$

These properties make sure that the market is free of arbitrage: Linearity requires $\wp(0) = 0$, which eliminates immediate arbitrage opportunities ($V_0 < 0$ and $V_t > 0$), and positivity removes the other two cases of arbitrage opportunities ($V_0 \leqslant 0$ and $V_t \gneq 0$)

1. The existence of pricing function is the fundamental assumption. It is vital such that we take it for granted. The intuitive interpretation of law of one price is the existence of pricing function: $X_1 = X_2 \implies \wp(X_1) = \wp(X_2)$
2. Some textbooks refer the law of one price to linearity of pricing function. However, we believe that it is the linear combination nature of portfolio that requires the linearity of pricing function.
   (a) Zero future value has zero price, $\wp(0) = 0$.

(b) Homogeneity (retail vs wholesale), $\wp(aX) = a\,\wp(X)$, is a special case of linearity.

3. Positivity is the characteristic of pricing function, any limited liability asset has a positive value now. In particular, $\wp(1) > 0$, thus for risk-free asset, the risk-free interest rate can be defined accordingly.

If $\wp(0) = 0$ and $\wp(1) > 0$ (special case of linearity and positivity respectively), the deterministic arbitrage opportunities are eliminated, the market is *absence of deterministic arbitrage* (special case of arbitrage opportunity).

Please note that if $\wp(X) = 0$, it is a zero-investment portfolio, we do not have $X = 0$ or even $\mathrm{E}(X) = 0$. For the zero-investment portfolio $(0, X)$

- When X is not random, $\wp(X) = 0 \implies X = 0$. However, a world without uncertainty is not interesting, the payoff X is a random variable for any non risk-free asset.
- When $X \neq 0$, it may not be an arbitrage opportunity: e.g., whenever there is a chance of $X > 0$, there is a chance of $X < 0$, there are possibilities of both future loss and future gain.
- When $\mathrm{E}(X) \neq 0$, it may fail to be an arbitrage opportunity. For even if $\mathrm{E}(X) > 0$, there may be a chance of $X < 0$.

By condition (1.15), a zero-investment portfolio is an arbitrage opportunity if and only if $X \gneq 0$.

B: Pricing Function

Payoffs modeled as random variables in financial market have finite expected returns. The marketable payoffs is a vector space spanned by primitive security, the pricing function is a map from the marketable payoffs to a set of price. By the Riesz representation theorem, for the payoff X at future time t, the pricing function is represented by an inner product

$$\wp(X) = \mathrm{E}(\Psi X) \qquad \Psi > 0 \tag{1.17}$$

the random variable Ψ is the stochastic discount factor, Ψ is positive because the pricing function is positive.

If the sample space is finite, $\Omega = \{w_1, w_2, \cdots, w_S\}$, let $x_i = X(w_i)$ and $\psi_i = \Psi(w_i)$, we have

$$X_0 = \wp(X) = \mathrm{E}(\Psi X) = \sum_{i=1}^{S} \psi_i x_i \mathrm{P}(w_i) = \sum_{i=1}^{S} g_i x_i = \mathbf{g}'\mathbf{x} \tag{1.18}$$

where the state prices are

$$g_i = \psi_i \mathrm{P}(w_i) > 0 \qquad i = 1, 2, \cdots, S$$

Let r be the risk-free interest rate, i.e., $1 = \wp(e^{rt})$, then

$$e^{-rt} = \wp(1) = \sum g_i$$

We have the risk-neutral pricing formula

$$X_0 = \wp(X) = e^{-rt}\,\mathrm{E}^Q(X) = e^{-rt} \sum_{i=1}^{S} x_i \mathrm{Q}(w_i) \tag{1.19}$$

with the risk neutral probabilities defined by

$$Q(w_i) = \frac{g_i}{\sum_{i=1}^{S} g_i} = e^{rt} g_i > 0$$

Obviously, there is $\sum Q(w_i) = 1$. We see that given the prices and payoffs of $S - 1$ risky assets, the risk neutral probabilities in Equation (1.19) are determined.

§ 1.4 Exercise

1.1 In Example 1.1.1, compute $Y_0 = aX_0 + b$, by find out a and b first.

1.2 Why do the actual probabilities not appear in pricing equation (1.4).

1.3 Why do we introduce the risk-neutral probability?

1.4 [group project] I flip a coin twice as in Example 1.1.1, the new proposition is game W: if both flips are tails, you get nothing, otherwise, you get \$9. How much is game W worth?

1.5 State the put-call parity, and prove it by way of pricing function.

1.6 Why free of arbitrage is essential in a financial market?

1.7 What is a self-financing portfolio?

1.8 State the definition of an arbitrage opportunity, and explain it.

1.9 Show that a self-financing portfolio with $V_0 \leqslant 0$ and $V_t > 0$ is an arbitrage opportunity.

1.10 In Example 1.1.2, if the price of the option is 4, is there an arbitrage opportunity? If yes, how to generate an arbitrage profit?

1.11 The following statements are equivalent: (a) The market is absence of arbitrage. (b) For any self-financing value process with $V_0 \leqslant 0$ and $V_t \geqslant 0$, we must have $V_t = V_0 = 0$.

1.12 When the market is absence of arbitrage, for any value process with $V_t - V_0 \geqslant 0$, we must have $V_t - V_0 = 0$. Is this statement true? Explain your answer.

1.13 Show that if there exist an arbitrage opportunity, such that $V_0 = 0$, and $V_t \gneqq 0$. Then we are able to construct a portfolio X with $X_0 = 0$, and $X_T \gneqq 0$ for any time $T > t$.

1.14 Show that any weak-positive combination ($h_i \geqslant 0$, and $\sum h_i > 0$, or $\mathbf{h} \gneqq 0$) of arbitrage opportunities is an arbitrage opportunity.

1.15 In Example 1.1.3, find o_2 that rule out arbitrage.

1.16 In Example 1.1.3, show that there does not exist risk-neutral probability measure.

1.17 Let $\wp(\cdot)$ be the pricing function and V_t be the value process of a self-financing portfolio. In a T-step discrete time model, show that

$$V_t = \wp_{t,T}(V_T) \qquad 0 \leqslant t \leqslant T$$

1.18 [group project] Is it possible that the game of fair play in Example 1.1.1 exists in real world? How to price game X?

§ 1.5 Appendix

We review some concepts in finance and mathematics. We collect proofs here and provide close related mathematical theorems for those who pursuit a rigorous foundation.

1.5.1 Derivatives

Whether you love derivatives or hate them, you cannot ignore them! A financial derivative is a contract that its value depends on the performance of other, more basic, underlying variables. A stock option, for example, is a derivative whose value is dependent on the price of a stock.

A: Option

A call option gives the holder of the option the right (but not the obligation) to buy a certain asset (underlying asset) by a certain date (expiration date, maturity) for a certain price (exercise price, strike price). A put option gives the holder the right to sell the underlying asset by the expiration date for the exercise price.

Remark: three "certain"; payoff of options

American options can be exercised at any time on or before the expiration date. A European-type option has the same terms as its American counterpart except that it cannot be surrendered (exercised) before the expiration date of the contract.

B: Forward and Futures Contract

A forward contract is an agreement to buy or sell an asset at a certain future time (delivery date) for a certain price (delivery price).

One of the parties to a forward contract assumes a long position and agrees to buy the underlying asset on a certain specified future date for a certain specified price. The other party assumes a short position and agrees to sell the asset on the same date for the same price. The value of a forward contract at the time it is first entered into is zero. (forward price is set according to such practice)

A futures contract is standardized forward contract, normally traded on an exchange with a mechanism that gives the two parties a guarantee that the contract will be honored

Remark: a futures contract or futures contracts, where the word *futures* is in plural form.

1.5.2 Linear Algebra

Tom buys 1 apple and 2 banana for \$4, and Jerry buys 3 apples and 1 banana for \$7, what are the costs of apple and banana each?

Let the cost of apple be x_1, and the cost of banana be x_2, then

$$x_1 + 2x_2 = 4$$
$$3x_1 + x_2 = 7$$

which can be written in matrix form

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

or view as vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

in row view, each row is a line (in $\mathbb{R}^2$, plane in $\mathbb{R}^3$), in column view, each column is a point in space.

It is easy to verify that $x_1 = 2$ and $x_2 = 1$.

A *linear combination* is an expression constructed from a group of terms by multiplying each term by a constant and adding the results. For an instance, let

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

then $x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2$ is a linear combination of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, where x_1 and x_2 are constants.

A: Linear Independence

Two vectors are linearly dependent if they are on the same line. Three vectors are linearly dependent when they lie on the same plane.

The vectors in a set $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_N\}$ are said to be *linearly independent* if the equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_N\boldsymbol{a}_N = 0$$

can only be satisfied by $x_i = 0$ for $i = 1, 2, \cdots, N$. This implies that no vector in the set can be represented as a linear combination of the remaining vectors in the set. In other words, a set of vectors is linearly independent if the only representations of 0 as a linear combination of its vectors is the trivial representation in which all the scalars x_i are zero.

Remark: the *span* of $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_N\}$ is the set of all linear combinations of the elements of $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_N\}$. Symbolically, $\text{sp}(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_N) = \{\sum_{i=1}^{N} x_i\boldsymbol{a}_i : [x_1; x_2; \cdots; x_N] \in \mathbb{R}^N\}$.

B: Rank

The *rank* of a matrix $\mathbf{A}$ is the dimension of the vector space generated (or spanned) by its columns. A matrix's rank is one of its most fundamental characteristics. There are multiple equivalent definitions of rank.

1. The maximum number of linearly independent rows or columns of $\mathbf{A}$.
2. The rank is also the dimension of the image of the linear transformation that is given by multiplication by $\mathbf{A}$.

C: Solution Set to Linear System

The vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_N\boldsymbol{a}_N = \mathbf{b}$$

is equivalent to a matrix equation of the form

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A}$ is an $M \times N$ matrix, $\mathbf{x}$ is a column vector with N entries, and $\mathbf{b}$ is a column vector with M entries.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_N \end{bmatrix}_{M\times N} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

A linear system $\mathbf{Ax} = \mathbf{b}$ may behave in any one of three possible ways: (N is the number of columns)

1. The system has no solution: $\operatorname{rank}([\mathbf{A}, \mathbf{b}]) \neq \operatorname{rank}(\mathbf{A})$
2. The system has a unique solution: $\iff \operatorname{rank}([\mathbf{A}, \mathbf{b}]) = \operatorname{rank}(\mathbf{A}) = N$
3. The system has infinitely many solutions: $\operatorname{rank}([\mathbf{A}, \mathbf{b}]) = \operatorname{rank}(\mathbf{A}) < N$

Remark: The system has a unique solution, if and only if the linear combination of column vectors will fill the whole space $\mathbb{R}^N$.

1.5.3 Real Analysis

A real number is a value that represents a quantity along a line. The real numbers include all the rational numbers and all the irrational numbers. A suitably rigorous definition of the real numbers is an axiomatic definition by a set of field axioms (properties on addition and multiplication), positivity axioms (properties for positive numbers) and completeness axiom (supremum).

A: Vector Space

Essentially like the set of real numbers, a *field* is a set of numbers possessing addition, subtraction, multiplication and division operations. The elements of a field are called *scalars*.

A *vector space*, also called a *linear space*, is a collection of objects called *vectors*, which may be added together and multiplied by scalars. A vector space is a set together with two binary operations: operations that combine two entities to yield a third, called vector addition and scalar multiplication. To qualify as a vector space, addition and multiplication have to adhere to a number of requirements called axioms (such as distributivity of scalar multiplication with respect to vector addition, and more).

B: Function and Operator

A *function* is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

- Monotonically increasing (also increasing or non-decreasing): if for all x and y such that $x \leqslant y$ one has $f(x) \leqslant f(y)$
- Strictly increasing: If the order $\leqslant$ in the definition of monotonicity is replaced by the strict order $<$. Functions that are strictly increasing are one-to-one.

An operator is a mapping from one vector space (or module) to another. From the point of view of functional analysis, calculus is the study of two linear operators: the differential operator, and the integral (antiderivative) operator. Operators are also involved in probability theory, such as expectation, which is a positive linear operator.

- A functional is an operator that maps a vector space to its underlying field.
- The most common kind of operator encountered are linear operators. Let U and V be vector spaces over a field F. Operator $\ell : \mathsf{U} \to \mathsf{V}$ is called linear if

$$\ell(aX + bY) = a\ell(X) + b\ell(Y)$$

for all X, Y in U and for all a, b in F. (e.g., $\mathbb{R}^M \to \mathbb{R}^N$: $\mathbf{y} = \ell(\mathbf{x}) = \mathbf{A}\mathbf{x}$)

In mathematics, the term linear function refers to some distinct notions: In calculus and related areas, a linear function is a polynomial function of degree zero or one, or is the zero polynomial

$$f(x) = ax + b$$

In linear algebra and functional analysis, a linear function is often referred to as a linear functional.

C: Riesz Representation Theorem

Given a real number $p > 1$, let $\mathsf{L}^p(\Omega, \mathcal{F}, \mathrm{P})$ be the collection of random variables such that $\mathrm{E}(|X|^p) < \infty$ for any random variable X. Define X to be payoff space of the market that has a finite number of assets with payoffs in L^p. Then for every linear functional $\wp : \mathsf{X} \to \mathbb{R}$, there exists a unique Ψ in L^q with $q = \frac{p}{p-1}$ such that

$$\wp(X) = \mathrm{E}(\Psi X)$$

for every $X \in \mathsf{X}$. (See Yeh (2014, P461, Theorem 18.7) for more discussion)

1.5.4 Proofs

The postulates on asset prices and other fundamental assumptions such as perfect market assumptions are the starting points to derive results in financial theories. For example, pricing functions should be linear and positive.

A: Zero Payoff Zero Price

Proof to Proposition 1.1: If the payoff of an asset is zero, the price must be zero. For any asset X, long asset X and the asset with zero payoff

$$\wp(X) = \wp(X + 0) = \wp(X) + \wp(0) \implies \wp(0) = 0$$

Method 2: Long 2 shares of zero-payoff asset

$$2\wp(0) = \wp(2 \cdot 0) = \wp(0) \implies \wp(0) = 0$$

B: FTAP

Proof to Theorem 1.3: In a finite market (not necessary finite state)

$\Longrightarrow$, if the linear pricing function is positive, then there is no arbitrage. For if there is $V_t \gneq 0$, then $V_0 = \wp(V_t) > 0$, never will be $V_0 \leqslant 0$. Which rules out zero-investment arbitrage opportunities and mixed arbitrage opportunities. Linearity eliminates immediate arbitrage opportunities. Thus, the three case of arbitrage opportunities are all ruled out, the market is free of arbitrage

$\Longleftarrow$: Will be deferred to later chapters. We will first prove it in binomial model (finite state, complete market), and then in the Black-Scholes market (continuous state, complete market), and finally in a general market setting (incomplete market).

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